Chapter 7
Mechanics

7.1 Tautochrone problem

7.1.1 Non-relativistic case

We start reviewing applications of fractional calculus with the following mechanical problem.

A classical, non-relativistic, point particle of mass m with potential energy $U = mgy$ begins to slide without friction along a curve running through a vertical plane $x-y$ to the origin and reaches it at time τ (Fig. 7.1). The problem is to find the function $\tau(h)$ that specifies the total time of descent from an initial height h. A special case of the problem when $\tau(h) = \text{const}$ is called the *tautochrone problem* (from Greek prefixes *tauto* meaning "same" and *chrono* "time").

The principle of conservation energy says

$$\frac{m}{2}\left(\frac{ds}{dt}\right)^2 = mg(h-y),$$

where s is the length of the curved segment between the origin and a current position, and $h-y$ is the descent of the particle during time t. Hence

$$dt = -\frac{ds}{\sqrt{2g(h-y)}}.$$

After integrating, we directly arrive at the equation

$$\tau(h) = \int_0^h \frac{ds/dy}{\sqrt{2g(h-y)}}dy = \sqrt{\frac{\pi}{2g}}\, {}_0^{1/2}\mathsf{D}_h s(h)$$

called *Abel's integral equation*, solution of which has opened a way to the land of fractional equations (Abel, 1881).

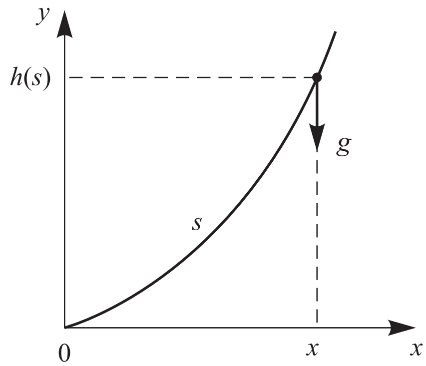

Fig. 7.1 Abel's mechanical problem.

Muñoz and Fernandéz-Anaya (2010) apply the fractional approach to investigation of the properties of the tautochrone and brachistochrone curves by introducing a family of curves complying with relations where the time of descent is proportional to a fractional power of the height difference.

7.1.2 Relativistic case

The relativistic counterpart of this problem has been studied by Kamath (1992). The methods of fractional calculus are shown to be more useful in the derivation of the exact relativistic tautochrone. Relativistic kinematics yields the equation of conservation energy

$$mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} + Q,$$

where Q is the energy lost by the particle from the gravitation field as it is released from a height h and is given by

$$Q = mc^2\left\{1 - \exp[g(h-y)/c^2]\right\}$$

(Goldstein and Bender, 1986). From two these equations, we find

$$v(y) = c\sqrt{1-\exp[2g(y-h)/c^2]}.$$

The time of fall is

$$T = \int_0^T dt = -\int_h^0 \frac{ds}{v(y)} = \int_0^h \frac{s'(y)dy}{c\sqrt{1-\exp[2g(y-h)/c^2]}},$$

with $s'(y) = ds/dy$ being the arclength along the path joining the initial $(x_0, y_0 = h)$ and final $(0,0)$ end points. By rewriting this equation as

$$cT = i\int_0^h \frac{e^{-gy/c^2}s'(y)dy}{\sqrt{e^{-2gh/c^2} - e^{-2gy/c^2}}} = -\frac{ic^2}{2g}\int_0^h \frac{(-2g/c^2)e^{-gy/c^2}}{\sqrt{e^{-2gh/c^2} - e^{2gy/c^2}}}s'(y)dy,$$

one arrives at the fractional equation for the function $\eta(h) = s'(h)e^{gh/c^2}$ determining the sought curve:

$$c\sqrt{\pi}\,{}_0\mathsf{D}_{\mu(h)}^{-1/2}\eta(h) = 2iTg,$$

with $\mu(h) = e^{2gh/c^2} - 1$. Converting the equation to

$${}_0\mathsf{D}_{\mu(h)}^{1/2}\,{}_0\mathsf{D}_{\mu(h)}^{-1/2}\eta(h) = {}_0\mathsf{D}_{\mu(h)}^{1/2}(2iTg/c\sqrt{\pi}),$$

and using the composition rules, the author reduces it to the form

$$\sqrt{\pi}e^{gh/c^2}s'(h) = \frac{2iTg}{c\sqrt{\pi}}\frac{d}{d\mu(h)}\int_0^h \frac{-2ge^{-2gy/c^2}dy}{c^2\sqrt{e^{-2gh/c^2} - e^{-2gy/c^2}}}$$

$$= 2\frac{2iTg}{c\sqrt{\pi}}\frac{d[\mu(h)]^{1/2}}{d\mu(h)} = \frac{2iTg}{c\sqrt{\pi}}[\mu(h)]^{-1/2}.$$

Solution of this equation leads to the following parametric representation of the sought tautochrone:

$$e^{2gy/c^2} = 1 + \left(\frac{2Tg}{\pi c}\right)^2 \cos^2\theta,$$

$$\frac{gx}{c^2} = \theta - \frac{\pi}{2} + a\left(\frac{\pi}{2} - \arctan\left(\frac{1}{a}\tan\theta\right)\right)$$

with

$$a = 1 + \left(\frac{2Tg}{\pi c}\right)^2.$$

The non-relativistic tautochrone problem was generalized to an arbitrary potential $U(y)$ as well (Gómez and Marquina, 2008). In this case

$$dt = -\frac{ds}{\sqrt{(2/m)U(h-y)}}$$

and

$$T = -\int_{y_0}^{y} \frac{s'(y)dy}{\sqrt{(2/m)[U(y_0) - U(y)]}}. \tag{7.1}$$

Making $z = U(y)$, one can write

$$\frac{ds}{dy}dy = \frac{ds}{dz}dz$$

with

$$dz = U'dy,$$

so then Eq. (7.1) takes the form

$$T = -\sqrt{\frac{m}{2}} \int_{z_0}^{z} \frac{s'(z)dz}{\sqrt{z_0 - z}}.$$

Solution of this equation

$$x = \int_0^y \sqrt{\frac{2T^2}{m\pi^2}\frac{U'^2}{U} - 1}\, dy$$

was found in (Flores and Osler, 1999).

7.2 Inverse problems

7.2.1 Finding potential from a period-energy dependence

Let us look at Sect. 12 of "Mechanics" (Landau and Lifshitz, 1981) devoted to finding of potential energy $U(x)$ from the oscillation period T given as a function of the total energy E, $T = T(E)$ (Fig. 7.2). Starting as before from the principle of energy conservation, they obtain the integral equation

$$T(E) = 2\sqrt{2m} \int_0^{X(E)} \frac{dx}{\sqrt{E - U(x)}}, \tag{7.2}$$

where $X(E)$ is a root of equation $U(x) = E$ (for simplicity let the potential be an even function, monotonically increasing with moving from the origin). Passage to integration variable U leads to the fractional differential equation

$$T(E) = 2\sqrt{2m} \int_0^E \frac{dX(U)}{dU} \frac{dU}{\sqrt{E - U}} = 2\sqrt{2m\pi}\ {}_0^{1/2}\mathsf{D}_E X(E),$$

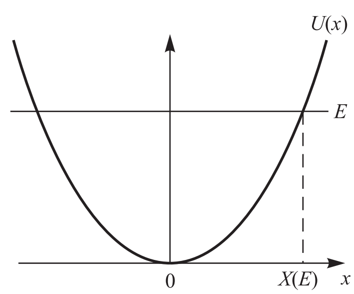

Fig. 7.2 Illustration to Eq. (7.2).

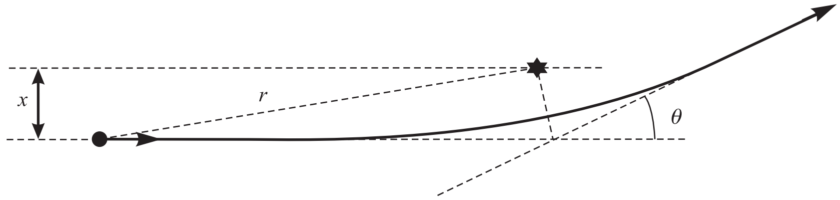

Fig. 7.3 Illustration to Eq. (7.3).

the integration of which yields Eq. (12.2) of the mentioned "Mechanics":

$$X(U) = \frac{1}{2\pi\sqrt{2m}} \int_0^U \frac{T(E)dE}{\sqrt{U-E}}.$$

It belongs to the class of *inverse problems*[1] formulated as finding properties of a system by observation of its motion. Just as Newton came to differential calculus considering a primal mechanical problem, Abel found a way to fractional differential calculus solving some special inverse problem.

7.2.2 Finding potential from scattering data

One more inverse mechanical problem is to find the interaction potential $U(r)$ from experimental data on scattering of particles with a given energy E. It is related to solving some integral equation (see (Buck, 1974) and Sect. 11.6 of book (Pavlenko, 2002)). The simplest statement of this problem is assuming the potential function $U(r)$ to be monotonically increasing and vanishing at infinity find it form the connection between the scattering angle θ and the impact parameter x is expressed by formula (Landau and Lifshitz, 1981) (Fig 7.3)

$$\theta(x) = \frac{x}{E} \int_x^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2 - x^2}}. \tag{7.3}$$

The formula represents a right-hand Gerasimov-Caputo semiderivative with respect to x^2:

$$\Theta(x) \equiv \frac{\theta(x)}{x} = \frac{1}{E} \int_{x^2}^\infty \frac{dU}{dr^2} \frac{dr^2}{\sqrt{r^2 - x^2}} = \frac{\sqrt{\pi}}{E}\, {}_{x^2}^{1/2}\mathsf{D}_\infty U(x^2).$$

Inverting this relation gives the result

$$U(r) = \frac{E}{\sqrt{\pi}}\, {}_{r^2}\mathsf{I}_\infty^{1/2} \Theta(r^2) = \frac{2E}{\pi} \int_r^\infty \frac{\theta(x)dx}{\sqrt{x^2 - r^2}}, \tag{7.4}$$

[1] The term "inverse problem" was introduced by Soviet-Armenian astrophysicist Victor Ambartsumian. His paper on the inverse Sturm-Liouville problem (1929) was found by Swedish mathematicians at the end of the Second World War and accepted as a foundation of a new discipline.

coincident with Eq. (11.16) in the book (Pavlenko, 2002). Finding $x(\theta)$ from the equation

$$\frac{1}{2}\frac{dx^2}{d\cos\theta} = \left[\frac{d\sigma(\theta)}{d\Omega}\right]_{\text{exper}},$$

inverting the obtained function and substituting the result $\theta(x) = \theta_{\text{exper}}(x)$ into Eq. (7.4), we come to the desired function $U(r)$.

7.2.3 Stellar systems

The construction of self-consistent models for stellar systems is of great interest in astrophysics. The most straightforward way to built such models is to start with an assumed potential defining the mass density ρ and the families of stars orbits with the distribution function f. The integral relation connecting f and ρ is known as the *self-gravitation equation*. Pedraza et al. (2008) generalized the classical methods involving the fractional derivatives. We sketch here this approach.

According to Jean's theorem, the phase space distribution function $f(\mathbf{r},\mathbf{v})$ is a function only of the isolating integrals of motion that are conserved in each orbit. For spherical symmetry these are the energy E and the angular momentum L_z.[2] Assume that Φ is the gravitational potential and define a relative potential $\Psi = -\Phi + \Phi_0$ and a relative energy $\varepsilon = -E + \Phi_0$, in such a way that the system has only stars with energy $\varepsilon > 0$. An axisymmetric system admits two isolated integrals: z-component of the angular momentum about the z-axis, $L_z = Rv_\varphi$ and $\varepsilon = -E + \Phi_0$. For a steady-state axisymmetric stellar system, the even part of the distribution function with respect to L_z, f_+, related to the mass density as

$$\rho(R,\Psi) = \frac{4\pi}{R}\int_0^{\Psi}\int_0^{R\sqrt{2(\Psi-\varepsilon)}} f_+(\varepsilon,L_z)dL_z d\varepsilon. \tag{7.5}$$

The authors seek the solution of the integral equation in the form

$$f_+(\varepsilon,L_z) = \sum_n L_z^{2\alpha_n} h_n(\varepsilon).$$

Inserting it into Eq. (7.5) and integrating with respect to L_z they obtain

$$\rho(R,\Psi) = \sum_n R^{2\alpha_n}\tilde{\rho}_n(\Psi), \quad \alpha_n > -1/2, \tag{7.6}$$

where

$$\tilde{\rho}_n(\Psi) = \frac{4\pi 2^{\alpha_n+1/2}}{2\alpha_n+1}\int_0^{\Psi} h_n(\varepsilon)(\Psi-\varepsilon)^{\alpha_n+1/2}d\varepsilon.$$

[2] As indicated in the cited work, as far back as Eddington showed that it is possible to obtain such distribution functions by first expressing the density as a function of the potential, and then solving an Abel integral equation.

This integral equation can easily be inverted by means of fractional derivatives method,

$$h_n(\varepsilon) = \frac{{}_0\mathrm{D}_{\Psi}^{\alpha_n+3/2}\tilde{\rho}_n(\Psi)\Big|_{\Psi=0}}{\pi 2^{\alpha_n+3/2}\Gamma(\alpha_n+1/2)},$$

so the result is

$$f_+(\varepsilon, L_z) = \sum_n \frac{{}_0\mathrm{D}_{\Psi}^{\alpha_n+3/2}\tilde{\rho}_n(\Psi)\Big|_{\Psi=0}}{\pi 2^{\alpha_n+3/2}\Gamma(\alpha_n+1/2)} L_z^{2\alpha_n}.$$

Here $\tilde{\rho}_n(\Psi)$ is assumed to be known because of Eq. (7.6).

7.3 Motion through a viscous fluid

7.3.1 Entrainment of fluid by a moving wall

We will consider motion of a body in an incompressible viscous Newton's fluid. In absence of external bulk forces, the unsteady flow of such a fluid is governed by the Navier-Stockes (N-S) equations system

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2},$$

where $u_i,\ i = 1,2,3$, are velocity components along x_i-coordinate axis respectively, ρ is the density, p is the pressure, and ν denotes the kinematic viscosity. This system should be supplemented with the incompressibility condition

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0.$$

For specificity of solution, this system must be accompanied by initial and boundary conditions.

Let us apply the N-S equations to the problem of entrainment of fluid by an unbounded (or of large size) plate moving in xOz-plane along x-axis (Fig. 7.4). The plate is at rest until $t = 0$ and then begins to move with a constant velocity V to positive direction of x-axis. The liquid particles velocity will contain only one nonzero component, $u_x = u$, and this component, due to unboundedness of the plate, will depend upon only z-coordinate,

$$u_x = u(z,t), \quad u_y = u_z = 0.$$

The pressure fall is zero too,

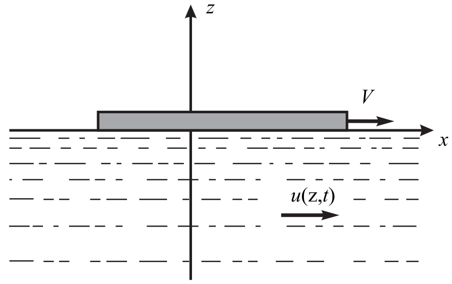

Fig. 7.4 To the problem of a plate on viscous fluid surface.

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0,$$

and we reduce the initial N-S system to the equation

$$\frac{\partial u(z,t)}{\partial t} = \nu \frac{\partial^2 u(z,t)}{\partial z^2} \tag{7.7}$$

of the diffusion type under following initial and boundary conditions:

$$u(z,t) = \begin{cases} 0, & \text{if } t = 0 \text{ and } z \leqslant 0; \\ V, & \text{if } t \geqslant 0 \text{ and } z = 0; \\ 0, & \text{if } t \geqslant 0 \text{ and } z = -\infty. \end{cases} \tag{7.8}$$

For solving the problem, we apply the Laplace transform method with respect to time. Multiplying both sides of Eq. (7.7) by $e^{-\lambda t}dt$, integrating over the positive half-axis,

$$\int_0^\infty e^{-\lambda t} \frac{\partial u(z,t)}{\partial t} dt = \nu \int_0^\infty e^{-\lambda t} \frac{\partial^2 u(z,t)}{\partial z^2} dt,$$

and applying the rule of integration by parts to the left-hand side,

$$\int_0^\infty e^{-\lambda t} \frac{\partial u(z,t)}{\partial t} dt = e^{-\lambda t} u(z,t) \Big|_0^\infty + \nu \int_0^\infty e^{-\lambda t} \frac{\partial^2 u(z,t)}{\partial z^2} dt = \nu \int_0^\infty e^{-\lambda t} \frac{\partial^2 u(z,t)}{\partial z^2} dt,$$

and the rule of commutation of integral and differential operations with respect to independent variables to the right-hand side,

$$\int_0^\infty e^{-\lambda t} \frac{\partial^2 u(z,t)}{\partial z^2} dt = \frac{\partial^2}{\partial z^2} \int_0^\infty e^{-\lambda t} u(z,t) dt,$$

we get the ordinary differential equation

$$\frac{d^2 \widehat{u}(z,\lambda)}{dz^2} - \frac{\lambda}{\nu} \widehat{u}(z,\lambda) = 0, \tag{7.9}$$

for the Laplace transform of the solution

$$\widehat{u}(z,\lambda)=\int_0^{\infty} e^{-\lambda t}u(z,t)dt.$$

The Latter obeys the conditions

$$\widehat{u}(z,\lambda)=\begin{cases} V/\lambda, & z=0; \\ 0, & z=-\infty; \end{cases} \tag{7.10}$$

following from the original boundary conditions (7.8). The general solution of Eq. (7.9) is represented as

$$\widehat{u}(z,\lambda)=Ae^{z\sqrt{\lambda/v}}+Be^{-z\sqrt{\lambda/v}}.$$

Upon using Eq. (7.10), we find

$$\widehat{u}(z,\lambda)=(V/\lambda)e^{z\sqrt{\lambda/v}}, \quad z<0.$$

Returning to the original $u(z,t)$ is possible by computing integral

$$u(z,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\widehat{u}(z,\lambda)d\lambda=\frac{V}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t+z\sqrt{\lambda/v}}\frac{d\lambda}{\lambda} \tag{7.11}$$

over a straight line in the complex plane parallel to imaginary axis and called the *convergence axis of the Laplace integral*. The integrand in (7.11) has a branchpoint at $\lambda=0$ so function must consider the infinite sequence of closed loops and use the Jordan lemma for computing the limit of corresponding integrals.

Let us draw a closed contour $ABCDEFA$, consisting of rectilinear segment AB, semicircle of a large radius R, two cuts CD and EF and a small circle DE of radius r around the origin (Fig. 7.5). The integrand has no singularities in the domain bounded by $ABCDEFA$, and according to the Cauchy theorem

$$I_{ABCDEFA}=\oint\limits_{ABCDEFA} e^{\lambda t+z\sqrt{\lambda/v}}\frac{d\lambda}{\lambda}=0$$

whence

$$\int_A^B e^{\lambda t+z\sqrt{\lambda/v}}\frac{d\lambda}{\lambda}\equiv I_{AB}=-I_{BC}-I_{CD}-I_{DE}-I_{EF}-I_{FA}.$$

Evidently, unlimited increasing R produces

$$I_{AB}\to\int_{\sigma_\infty}^{\sigma+\infty} e^{\lambda t+z\sqrt{\lambda/v}}\frac{d\lambda}{\lambda}, \quad I_{BC}\to 0, \quad I_{FA}\to 0,$$

$$I_{DE}=-2\pi i\operatorname*{Res}_{\lambda=0}\left\{\frac{e^{\lambda t+z\sqrt{\lambda/v}}}{\lambda}\right\}=-2\pi i.$$

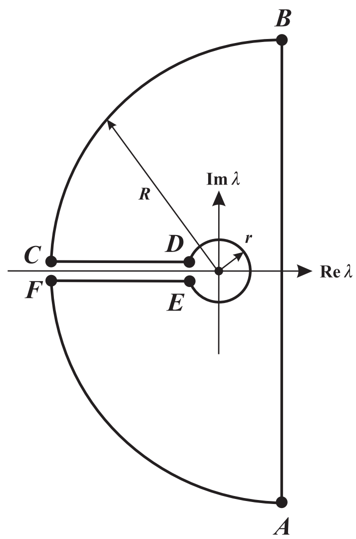

Fig. 7.5 The integration contour *ABCDEFA*.

Taking for cut CD

$$\lambda = \xi^2 e^{i\pi} = -\xi^2, \quad \sqrt{\lambda} = \xi e^{i\pi/2} = i\xi, \quad d\lambda = -2\xi d\xi,$$

and for cut EF

$$\lambda = \xi^2 e^{-i\pi} = -\xi^2, \quad \sqrt{\lambda} = \xi e^{-i\pi/2} = -i\xi, \quad d\lambda = -2\xi d\xi,$$

we obtain

$$I_{CD} \to \frac{2}{i}\int_0^\infty e^{-\xi^2 t - i(z/\sqrt{\nu})\xi}\frac{d\xi}{\xi},$$

and

$$I_{EF} \to \frac{2}{i}\int_0^\infty e^{-\xi^2 t + i(z/\sqrt{\nu})\xi}\frac{d\xi}{\xi},$$

respectively. As a result, we have

$$g(z,t) := \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t + z\sqrt{\lambda/\nu}}\frac{d\lambda}{\lambda} = 1 + \frac{2}{\pi}\int_0^\infty e^{-t\xi^2}\sin\left(\frac{z\xi}{\sqrt{\nu}}\right)\frac{d\xi}{\xi} \qquad (7.12)$$

and consequently

$$u(z,t) = g(z,t)V.$$

Assuming that an acceleration of the plate is not zero after $t = 0$ but can be broken up into a series of small step changes in the velocity, the velocity of entrained fluid can be written in the form of *Duhamel's principle*:

$$u(y,t) = g(y,t)V(0) + \int_0^t g(y,t-t')\dot{V}(t')dt'.$$

Its physical meaning is evident: perturbations added to the velocity field by changes in the plate velocity spread according to diffusion law, and the shift of time in the integral term reflects the retardation effect. From mathematical point of view, $g(z,t)$ is the Green function of the equation under consideration.

If the initial moment is not zero but, say, a, the formula becomes

$$u(y,t) = g(y,t-a)V(a) + \int_a^t g(y,t-t')\dot{V}(t')dt',$$

with admissible value $a = -\infty$. The viscous force per unit surface of the plate (the shear stress) is represented by

$$\tau = \nu\rho \left(\frac{\partial u(y,t)}{\partial y}\right)_{y=0} \equiv \nu\rho u'(0,t)$$

$$= \nu\rho \left[g'(0,t-a)V(a) + \int_a^t g'(0,t-t')\dot{V}(t')dt'\right],$$

where

$$g'(0,t) = \left.\frac{\partial g(y,t)}{\partial y}\right|_{y=0} = \left[\frac{\partial}{\partial y}\left(1 - \frac{2}{\pi}\int_0^\infty e^{-(t-a)\xi^2} \sin\left(\frac{y\xi}{\sqrt{\nu}}\right)\frac{d\xi}{\xi}\right)\right]_{y=0}$$

$$= -\rho\sqrt{\frac{\nu}{\pi(t-a)}}.$$

Thus we see that

$$\tau = -\rho\sqrt{\frac{\nu}{\pi}}\left[\frac{V(a)}{\sqrt{t-a}} + \int_a^t \frac{\dot{V}(t')dt'}{\sqrt{t-t'}}\right]$$

$$= -\rho\sqrt{\nu}\left[\frac{V(a)}{\sqrt{\pi(t-a)}} + {}_a^{1/2}\mathsf{D}_t V(t)\right] = -\rho\sqrt{\nu}\, {}_a\mathsf{D}_t^{1/2}V(t). \tag{7.13}$$

Eq. (7.13) shows that the viscous stress at the plate at time t depends on all preceding ($t' < t$) motion of the wall, or, at greater length, the stress observed at time t at a point (x,y) is a result of contributions of liquid particles coming from other points (x',y) of the layer, where they were located, say, at $t' < t$. Because of the translation invariance of the solution relative to x, the same velocity distribution at the moment t' took place also in the observing point (x,y), and the space-shift $u(x',t) \mapsto u(x,t)$ is equivalent to the time-shift $u(x,t') \mapsto u(x,t)$. This is the simplest, mechanism of heredity, the "mechanical" memory. It was discussed in works (Slyoskin, 1955; Podlubny, 1999; Kulish and Lage, 2002) and others.

7.3.2 Newton's equation with fractional term

As follows from the foregoing, the equation connecting the force F acting on a thin plate of mass m_0 and surface area S immersed into a liquid with density ρ and kinematic viscosity ν, and the corresponding acceleration of this plate has the form

$$m_0 \frac{dV}{dt} + 2\rho S\sqrt{\nu}\, {}_a\mathsf{D}_t^{1/2} V(t) = F(t) \tag{7.14}$$

(the factor 2 accounts that both surfaces of the plate are in contact with the fluid).

In case of a sphere of mass m_0 and of radius R steadily moving through an incompressible viscous fluid under action of an external force $F(t)$, the motion equation has the form

$$m_0 \frac{dV}{dt} = F(t) + R(t),$$

where R is the resistance force:

$$R(t) = -6\pi\rho\nu RV(t) - \frac{2}{3}\pi\rho R^3 \frac{dV}{dt} - 6\pi\rho R^2\sqrt{\nu}\left[\frac{V(a)}{\sqrt{\pi(t-a)}} + \frac{1}{\sqrt{\pi}}\int_a^t \frac{\dot{V}\,d\tau}{\sqrt{t-\tau}}\right].$$

The first term represents the *Stokes formula* for the friction force as it appears in hydrodynamic consideration of a sphere motion through a viscous fluid; the second gives the force due to the mass of the liquid displaced by the particle (*associated mass*), and the third term called the *Boussinesq-Basset formula* accounts the retardation effect on the motion of the sphere due to the penetration depth of viscous unsteady flow around the sphere (Boussinesq, 1885; Basset, 1888). This formula has been derived on assumption that the sphere starts its motion in a motionless fluid. This means that the body itself has been at rest for a long time before a, so all perturbations excited by its preceding motion have vanished. In other words, the velocity $V(t)$ is 0 for all $t < a$, and the motion equation for a body of mass m_0 inserted into a fluid with kinematic viscosity ν can be written as

$$m\frac{dV}{dt} + b\, {}_a\mathsf{D}_t^{\alpha} V(t) + cV(t) = F(t), \quad V(t) \equiv 0 \quad \text{for} \quad t < a, \tag{7.15}$$

where $\alpha = 1/2$, coefficients

$$m = m_0, \quad b = 2\rho S\sqrt{\nu}, \quad c = 0$$

in case of a thin plate, and

$$m = m_0 + \frac{2}{3}\pi\rho R^3, \quad b = 6\pi\rho\sqrt{\nu}R^2, \quad c = 6\pi\rho\nu R$$

in case of a sphere. Finally, one can suppose that a wide range of problems related to motion of bodies through viscous liquids can be considered on the base of Eq. (7.14), that is the Newton equation with a fractional term, or shorter, the *frac-*

tional Newton equation (FNE). The value $\alpha = 1/2$ means that we deal with the Newtonian fluid. We leave the symbol α in this equation keeping in mind to use it for non-Newtonian fluids when $\alpha \neq 1/2$.

When the driving force F depends on the body coordinate, $x(t)$, the motion is governed by the fractional Newton equation

$$m\frac{dx^2}{dt^2} + b\,{}_a\mathsf{D}_t^{\alpha+1}x(t) + c\frac{dx}{dt} = F(x,t), \quad 0 < \alpha < 1. \tag{7.16}$$

As follows from Tauberian theorem, the main asymptotic (as $t \to \infty$) part of the solution $V^{\mathrm{as}}(t)$ satisfies an equation with a single fractional order derivative,

$$b\,{}_a\mathsf{D}_t^{\alpha}V^{\mathrm{as}}(t) + cV^{\mathrm{as}}(t) = F(t).$$

In what follows, we call it the *reduced fractional Newton's equation* (RFNE) in order to distinguish it from the *total fractional Newton equation* (TFNE) (7.15). The same terms will be used for coordinate representation of the motion equations (7.16).

7.3.3 Solution by the Laplace transform method

One of the first solutions of FNE, as I know, belongs to Russian mechanician Slyoskin (1955). Let us follow his way. Multiplying both sides of TFNE (7.14), rewritten as

$$m\frac{dV(t)}{dt} + b\left[\frac{V(0)}{\sqrt{\pi t}} + \frac{1}{\sqrt{\pi}}\int_0^{\infty}\frac{\dot{V}(t')dt'}{\sqrt{t-t'}}\right] = F(t),$$

by $e^{-\lambda t}dt$ and integrating from zero to infinity, he obtained the expression

$$m\lambda\widehat{V}(\lambda) - mV(0) + b\left[\frac{V(0)}{\sqrt{\lambda}} + \frac{1}{\sqrt{\pi}}\int_0^{\infty}dte^{-\lambda t}\int_0^t\frac{\dot{V}(t')dt'}{\sqrt{t-t'}}\right] = \widehat{F}(\lambda)$$

which after applying the Dirichlet rule to the last term in the left-hand side is reduced to the form

$$m\lambda\widehat{V}(\lambda) - mV(0) + b\left[\frac{V(0)}{\sqrt{\lambda}} + \frac{1}{\sqrt{\pi}}\left(\sqrt{\lambda\pi}\widehat{V}(\lambda) - \sqrt{\frac{\pi}{\lambda}}V(0)\right)\right] = \widehat{F}(\lambda).$$

Solution of this algebraic equation, representing the Laplace transform of the velocity of the plate, immersed into viscous fluid,

$$\widehat{V}(\lambda) = \frac{\widehat{F}(\lambda) + mV(0)}{\lambda(m + b/\sqrt{\lambda})}, \tag{7.17}$$

coincides with that found by Slyoskin (Eq. (2.30) in the cited book). He used this result for analyzing processes of slowing down of a plate in a viscous fluid, generating vortex sheets, moving the fluid between two plates and through a pipe, rotating in a cylinder, and others. Referring to Tauberian theorem, which claims that for any $f(t)$.

$$f(t) \sim At^{-\alpha},\ t \to \infty \quad \Leftrightarrow \quad \widehat{f}(\lambda) \sim A\Gamma(1-\alpha)\lambda^{\alpha-1},\ \lambda \to 0,$$

we bring here two asymptotic results.

1. The plate has the initial velocity V_0 and moves in a fluid without external force, that is $F = 0$. In this case,

$$\widehat{V}(\lambda) = \frac{mV_0}{\lambda(m+b/\sqrt{\lambda})} \sim \frac{mV_0}{b}\lambda^{-1/2}, \quad \lambda \to 0,$$

   therefore

$$V_1(t) \sim \frac{mV_0}{b\sqrt{\pi}}\,t^{-1/2}, \quad t \to \infty.$$

2. The plate being at rest till $t = 0$ undergoes action of a constant force F_0, accelerating the plate. Now, $V(0) = 0$, $F(t) = F_0$, and we obtain

$$V_2(t) \sim \frac{2F_0}{b\sqrt{\pi}}\,t^{1/2}, \quad t \to \infty.$$

7.3.4 Solution by the Green functions method

Representing Eq. (7.17) in the form

$$\widehat{V}(\lambda) = V(0)\widehat{G}(\lambda) + \frac{1}{m}\hat{F}(\lambda)\widehat{G}(\lambda),$$

with

$$\widehat{G}(\lambda) = \frac{\lambda^{1/2-1}}{\lambda^{1/2}+b/m}.$$

We identify the latter expression with the Laplace transform of the Mittag-Leffler function

$$G(t) = E_{1/2}(-(b/m)\sqrt{t}) \equiv e^{(b/m)^2 t}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{(b/m)\sqrt{t}} e^{-\xi^2}d\xi\right).$$

Consequently,

$$V(t) = V(0)G(t) + \frac{1}{m}\int_0^t G(t-t')F(t')dt'$$

with Green's function $G(t)$ given by the previous expression. Using this representation, one can investigate short-time behavior of the plate immersed into a viscous incompressible fluid. In particular, for the two problems whose long-time asymptotical solutions are found above, we obtain in a short-time domain ($t \ll (b/m)^2$):

$$V_1(t) \approx V_0 \left[1 - \frac{2}{\sqrt{\pi}} \left(\frac{b}{m}\right) t^{1/2}\right]$$

and

$$V_2(t) \approx \frac{F_0 t}{m} \left[1 - \frac{4}{3\sqrt{\pi}} \left(\frac{b}{m}\right) t^{1/2}\right].$$

Solution of a more general case of FNE, relating to a sphere moving through a non-Newtonian fluids,

$$m\frac{dV}{dt} + b\,{}_0\mathsf{D}_t^{\alpha} V(t) + cV(t) = F(t), \quad \alpha \in (0,1], \tag{7.18}$$

can be represented by means of Green's function found by Podlubny(1999) (see Sect. 5.1.9). It is expressed through the two-parameter Mittag-Leffler function and has the form

$$G(t) = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{c}{m}\right)^k t^k E^{(k)}_{1-\alpha,1+\alpha k}\left[-(b/m)t^{1-\alpha}\right].$$

7.3.5 Fractionalized fall process

Reading numerous articles with various applications of fractional differential calculus, we often meet a simple method of introducing fractional derivative into consideration. Namely, the author of such an article takes a well-known differential equation derived from first physical principles and says: "Now, we replace the first (second) derivative in this equation by its fractional counterpart". After this, he describes a method of solution of the equation, demonstrates numerical results, stress on its difference from the classical analogs and shows that they reduce to classical results when the order of the fractional derivative becomes integer. At the end of the articles, one can read that these results may be applied to some problems arising in viscoelasticity phenomena, physics of polymers or so.

Going this way, we might say, for example: let us take the Newton equation of motion

$$m\dot{V} = F$$

and consider its fractional generalization

$$m\,{}_0\mathsf{D}_t^{\alpha} V = F. \tag{7.19}$$

For notational convenience, we will use for such procedure the term *fractionalization* and call the resulting equation the *fractionalized equation*. Thus, we can say, that Eq. (7.19) is obtained by fractionalization of the classic Newton equation and for this reason may be called the *fractionalized Newton equation*, but we can't use the term for Eq. (7.18), because it is *derived* from the classic mechanic-hydrodynamic system of equations without replacing integer-order derivatives by fractional-order ones.

As an example of fractionalization, one can indicate the work (Fa, 2005) devoted to the problem of a body fall under action of homogeneous gravity and friction force proportional to its velocity. The Newton equation of motion was fractionalized by using the Caputo-Gerasimov time-derivative

$$m_\nu \, {}_0^\nu \mathsf{D}_t V + bV = mg.$$

Beginning with the case when the velocity term is absent,

$$m_\nu \, {}_0^\nu \mathsf{D}_t V = mg,$$

the author comes to the solution

$$V(t) = V_0 + \frac{mgt^\nu}{m_\nu \Gamma(1+\nu)}, \tag{7.20}$$

and obtains by additional integration

$$x(t) = \int_0^t V(\tau)d\tau = x_0 + V_0 t + \frac{mgt^{\nu+1}}{m_\nu \Gamma(2+\nu)}, \tag{7.21}$$

where V_0 is the initial velocity and x_0 the initial coordinate (the x-axis is down-directed). One can see that for short times the fractional derivative gives a greater velocity than the first one ($t^\nu > t$ for $\nu < 1$), while for long times, the velocity is smaller than in a classical case.

Evidently, the parameter m_ν for $\nu \neq 1$ does not have the dimension of mass. In (Narahari Achar et al., 2001; Ryabov and Puzenko, 2002), the generalized momentum is defined by

$$p_\nu = m_\nu \, {}_0^\alpha \mathsf{D}_t x(t), \quad \alpha = (1+\nu)/2,$$

in order to agree $p_1^2/(2m_1)$ with the dimension of energy. As a result, the total energy of the body is written by

$$E = \frac{p_\nu^2}{2m_\nu} + U(x) = \frac{m_\nu}{2}\left[{}_0^\alpha \mathsf{D}_t x(t)\right]^2 - mgx. \tag{7.22}$$

Observe that the rate of total energy (Fa, 2005) is expressed as

$$\frac{dE}{dt} = \left[\frac{m_\nu \beta V_0 t^{-\nu}}{\Gamma^2(1+\beta)} + \frac{mg}{\Gamma(1+\alpha)\Gamma(1+\beta)} - mg\right] V_0 + \frac{(mg)^2}{m_\nu}\left[\frac{\alpha}{\Gamma^2(1+\alpha)} - \frac{1}{\Gamma(2\alpha)}\right] t^\nu,$$

where

$$\alpha = \frac{1+\nu}{2}, \quad \beta = \frac{1-\nu}{2}.$$

Note that the kinetic energy increases and the velocity of full body's energy increment isn't proportional to the velocity of its motion, except cases when the body was motionless at initial instant

$$\frac{dE}{dt} = \frac{m^2 g^2}{m_\beta} \left[\frac{1}{\alpha[\Gamma(\alpha)]^2} - \frac{1}{\Gamma(2\alpha)} \right] t^\nu, \quad V_0 = 0.$$

If $\nu \to 1$ and $m_1 = m$, formulas (7.20), (7.21), (7.22) turn into well-known equations of classical mechanics:

$$V(t) = V_0 + gt,$$

$$x(t) = x_0 + V_0 t + \frac{gt^2}{2},$$

$$E = \frac{mV^2}{2} - mgx = \text{const}.$$

Accounting of resistive force leads to the next expressions for the velocity and coordinate of the body falling accordingly to fractional differential law

$$V(t) = V_0 E_{\nu,1}[-(b/m_\nu)t^\nu] + \frac{mg}{m_\nu} t^\nu E_{\nu,1+\nu}[-(b/m_\nu)t^\nu],$$

and

$$x(t) = x_0 + V_0 t E_{\nu,2}[-(b/m_\nu)t^\nu] + \frac{mg}{m_\nu} t^{1+\nu} E_{\nu,2+\nu}[-(b/m_\nu)t^\nu].$$

Having analyzed experiments on the six men falling from the height of 31 400 foots, author of the article comes to the conclusion that the fractional differential approach with $\beta = 0.998$ and $m/m_\beta = 1.457$ is not in worse agreement with the experiment dependencies $x(t)$ then the classical one. The author did not comment the cause of the fact.

Baleanu et al. (2010) representing a new version of the fractional Newtonian mechanics define the *fractional velocity* and *fractional momentum* for interval $[a,b]$ as

$$V(t) = (1/2)(\rho_1 \, {}_a^\alpha \mathsf{D}_t + \rho_2 \, {}_t^\beta \mathsf{D}_b) x(t),$$

and

$$p(t) = (m/2)(\rho_1 \, {}_a^\alpha \mathsf{D}_t + \rho_2 \, {}_t^\beta \mathsf{D}_b) x(t) = p_\alpha + p_\beta,$$

respectively, where $0 < \alpha, \ \beta \leqslant 1$, and dimensions of the constants ρ_1 and ρ_2 are $T^{\alpha-1}$ and $T^{\beta-1}$, respectively. The fractional version of Newton's second law is then written as the expression

$$(1/2)(\kappa_1 \, {}_t\mathsf{D}_b^\alpha p_\alpha + \kappa_2 \, {}_a\mathsf{D}_b^\beta p_\beta) = F, \tag{7.23}$$

added by transversality condition

$$\left[{}_t\mathsf{D}_b^{\alpha-1}p_\alpha - {}_a\mathsf{D}_t^{\beta-1}p_\beta\right]_a^b = 0. \tag{7.24}$$

The authors note that for $\alpha = 1,\ \beta = 1$ they have

$${}_a\mathsf{D}_t^\alpha = {}_a^\alpha\mathsf{D}_t = {}_t\mathsf{D}_b^\alpha = {}_t^\alpha\mathsf{D}_b = \frac{d}{dt},$$

and Eq. (7.24) reduces to the standard Newtonian equation. This is really so.

However, their next Remark 3.1, "If the generalized force in Eq. (7.23) is zero, then we can generalize Newton's first law as $\kappa_1\, {}_t\mathsf{D}_b^\alpha p_\alpha + \kappa_2\, {}_a\mathsf{D}_t^\beta p_\beta = 0$" seems to be non quite understable. Really, the first Newtonian law should not be considered as a plain mathematical sequence of the second one (in this case, it would be absolutely unnecessary in the system of mechanical axioms). In classical mechanics, the first law singles out from all possible coordinate systems, the family of inertial systems for which the second law is valid. From this point of view, the concepts of fractional velocity and fractional momentum look too artificial.

7.4 Fractional oscillations

7.4.1 Fractionalized harmonic oscillator

The harmonic oscillator, one of the simplest mechanical systems, whose motion is governed by a second-order being linear differential equation with constant coefficients and whose analogs are ubiquitous in physics, needs no introduction. From mathematical point of view, fractional generalizations of this system have been studied by Bagley and Torvik (1984), Gorenflo and Rutman (1995), Mainardi (1996), Blank (1997), Gorenflo and Mainardi (1997), Podlubny (1999), Narahary Achar et al. (2001), Trinks and Ruge (2002), Stanislavsky (2004), and other authors.

Fractionalization of a free oscillation process, governed by differential equation

$$\ddot{x}(t) + \omega^2 x(t) = 0, \quad \omega^2 = k/m, \quad x(0) = x_0, \quad \dot{x}(0) = V_0,$$

or, equivalently, by integral equation (Arfken and Weber, 1995)

$$x(t) = x_0 + V_0 t - \omega^2 \int_0^t d\tau \int_0^\tau x(t')dt',$$

can be performed by using both these formulations. The first way leads (for $\alpha \in (1,2]$) to equation

$${}_0\mathsf{D}_t^\alpha[x(t) - x_0 - V_0 t] + \omega^\alpha x = 0$$

with the R-L derivative or to

$${}_0^\alpha\mathsf{D}_t x(t) + \omega^\alpha x(t) = 0$$

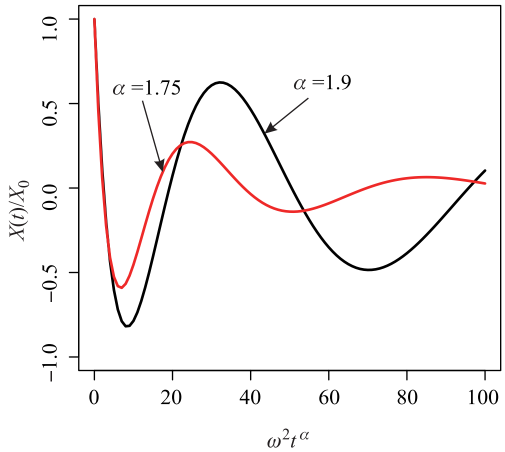

Fig. 7.6 Displacement as a function of time (in reduced variables) as per Eq. (7.25).

with the G-C derivative (Gorenflo and Rutman, 1995). The second way yields

$$x(t) = x_0 + V_0 t - \omega^{\alpha}\ {}_0\mathsf{I}_t^{\alpha} x(t)$$

(Narahary Achar et al., 2001). The solution, obtained by the Laplace transform method and expressed in terms of two-parameter Mittag-Leffler functions

$$x(t) = x_0 E_{\alpha,1}[-(\omega t)^{\alpha}] + V_0 t E_{\alpha,2}[-(\omega t)^{\alpha}]$$

and in case of $V_0 = 0$, has the form

$$x(t) = x_0 E_{\alpha,1}[-(\omega t)^{\alpha}]. \tag{7.25}$$

According to Gorenflo and Mainardi (1997), it can also be written as

$$x(t) = \frac{x_0}{\pi} \int_0^{\infty} \frac{\xi^{\alpha-1}\sin(\alpha\pi)}{\xi^{2\alpha} + 2\xi^{\alpha}\cos(\alpha\pi) + 1} e^{-\omega t \xi} d\xi.$$

Corresponding graph is plotted in Fig. 7.6.

As the authors write, their principal result is represented by expressions for the momentum and the total energy of the fractionalized oscillator. Following the fractionalization idea, they introduced these characteristics by generalization of corresponding classical ones:

$$p = m\mathsf{D}_t x(t) \ \mapsto\ p = m\ {}_0^{\alpha/2}\mathsf{D}_t x(t)$$

and

$$E = \frac{1}{2}kx^2 + \frac{1}{2}m[\mathsf{D}_t x(t)]^2 \mapsto\ E = \frac{1}{2}kx^2 + \frac{1}{2}m[\ {}_0^{\alpha/2}\mathsf{D}_t x(t)]^2,$$

where $1 < \alpha \leqslant 2$. As a result, in case $V_0 = 0$,

$$p = -m x_0 \omega^{\alpha} t^{\alpha/2} E_{\alpha,1+\alpha/2}[-(\omega t)^{\alpha}] \tag{7.26}$$

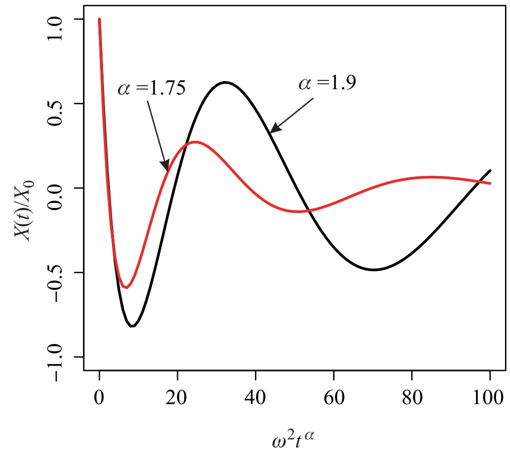

Fig. 7.7 Phase plane diagram (in reduced variables) for $\alpha = 1.9$.

and

$$E = \frac{1}{2}kx_0^2E_{\alpha,1}^2[-(\omega t)^\alpha] + \frac{1}{2}mx_0^2(\omega^2 t)^\alpha E_{\alpha,1+\alpha/2}^2[-(\omega t)^\alpha].$$

The phase diagram corresponding to Eqs. (7.25)–(7.26) is shown in Fig. 7.7.

In case of a simple harmonic oscillator problem, $\alpha = 2$,

$$x(t) = x_0 \sum_{m=0}^{\infty} \frac{[-(\omega t)^2]^m}{(2m)!} = x_0 \cos(\omega t),$$

the motion is periodic, the total energy is constant and the phase plane diagram is a closed elliptic curve. When $\alpha < 2$, the oscillator behavior changes: its response function decays and its diagram in the phase plane (x, p) is not a closed curve anymore but a logarithmic spiral similar to the diagram of the damping classical oscillator (see Figs.1 and 2 in (Narahari Achar et al., 2001)).

The next article of Narahari Achar et al. (2002) contains results of studying of the response and resonance characteristics of the fractional oscillator for several cases of forcing function. The inclusion of an external driving force $F(t)$ under zeroth initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0,$$

leads to the integral fractional equation

$$x(t) = {}_0\mathsf{I}_t^\alpha[-\omega_0^\alpha x(t) + F(t)],$$

solution of which for $\alpha \in (1, 2)$ is given by

$$x(t) = \int_0^t E_{\alpha,\alpha}[-\omega_0^\alpha (t-\tau)^\alpha](t-\tau)^{\alpha-1}F(\tau)d\tau.$$

Three special results are shown in Table 7.1, where

$$A_1 = \frac{1}{\sqrt{\omega_0^{2\alpha} + \omega^{2\alpha} + 2\omega_0^\alpha \omega^\alpha \cos(\alpha\pi/2)}},$$

$$\delta = \arctan\left[\frac{\omega^\alpha \sin(\alpha\pi/2)}{\omega_0^\alpha + \omega^\alpha \cos(\alpha\pi/2)}\right],$$

$$A_2 = \frac{2\omega}{\alpha\omega_0^{\alpha-1}\sqrt{\omega_0^4 + \omega^4 + 2\omega_0^2\omega^2\cos(2\pi/\alpha)}},$$

$$\beta = -\omega_0 \cos(\pi/\alpha),$$

$$\phi = \arctan\left[\frac{\omega_0^2 \sin((1+\alpha)\pi/\alpha) - \omega^2 \sin((1-\alpha)\pi/\alpha)}{\omega_0^2 \cos((1+\alpha)\pi/\alpha) + \omega^2 \cos((1-\alpha)\pi/\alpha)}\right],$$

and

$$K_\alpha(s) = \frac{\omega \sin(\pi\alpha)}{\pi(s^2+\omega^2)(s^{2\alpha} + 2s^\alpha \omega_0^\alpha \cos\pi\alpha + \omega_0^{2\alpha})}.$$

Table 7.1 Forcing and response functions

Forcing function $F(t)$	Response function $(x(t))$
$\delta(t)$	$t^{\alpha-1}E_{\alpha,\alpha}[-(\omega_0 t)^\alpha]$
$1(t)$	$t^\alpha E_{\alpha,\alpha+1}[-(\omega_0 t)^\alpha]$
$\sin(\omega t)$	$A_1\sin(\omega t-\delta) + A_2\exp(-\beta t)\cos[\omega_0 t\sin(\pi/\alpha)-\phi]$
	$+\int_0^\infty \exp(-st)K_\alpha(s)ds$

The response to the δ-function is shown in Fig. 7.8.

The latter response given in Table. 7.1 is obtained from the initial expression by using the Mittag-Leffler representation of the sine function

$$\sin(\omega\tau) = \omega\tau E_{2,2}[-(\omega\tau)^2]$$

and evaluating the Laplace transform of the resulting function

$$\widehat{x}(\lambda) = \omega\frac{\lambda^{-\alpha}}{1+(\omega_0/\lambda)^\alpha}\frac{\lambda^{-2}}{1+(\omega/\lambda)^2}.$$

Performing inverse transformation represents the responce function as a sum of three terms

$$x(t) = \frac{1}{2\pi i}\int_{\mathrm{Br}} \frac{\omega\exp(\lambda t)d\lambda}{(\lambda^2+\omega^2)(\lambda^\alpha+\omega_0^\alpha)} = x_1(t)+x_2(t)+x_3(t).$$

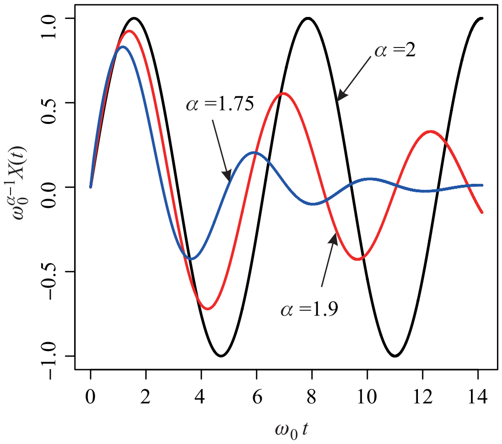

Fig. 7.8 Response function for a δ-forcing function for different values of α.

The first of them results from calculating the residues of the poles $\lambda = \pm i\omega$:

$$x_1(t) = \left[\frac{\omega_0^\alpha \sin(\omega t) + \omega^\alpha \sin(\omega t - \alpha\pi/2)}{\omega_0^{2\alpha} + \omega^{2\alpha} + 2\omega_0^\alpha \omega^\alpha \cos(\alpha\pi/2)}\right].$$

The second is defined by the residues for two other poles $\lambda = \omega_0 \exp(\pm i\pi/\alpha)$:

$$x_2(t) = \left[\frac{\omega e^{\lambda t}}{(\lambda^2 + \omega^2)(d/d\lambda)(\lambda^\alpha + \omega_0^\alpha)}\right]_{\lambda=\omega_0 \exp(\pm i\pi/\alpha)}$$

$$= \frac{2\exp[\omega_0 t \cos(\pi/\alpha)]C}{\alpha\omega_0^{\alpha-1}\left[\omega^4 + \omega_0^4 + 2\omega^2\omega_0^2 \cos(2\pi/\alpha)\right]},$$

with

$$C = \omega\{\omega_0^2 \cos[\omega_0 t \sin(\pi/\alpha) - (1+\alpha)\pi/\alpha]$$
$$+\omega^2 \cos[\omega_0 t \sin(\pi/\alpha) + (1-\alpha)\pi/\alpha]\}.$$

After some algebra, these terms are reduced to the form given in the Table 7.1:

$$x_1(t) = A_1 \sin(\omega t - \delta),$$

$$x_2(t) = A_2 e^{-\beta t} \cos[\omega_0 t \sin(\pi/\alpha) - \phi].$$

The third integral term originates from the loop consisting of the small circle of radius r and the two lines CD and EF parallel to the negative real axis (see Fig. 7.5)

$$x_3(t) = -\frac{1}{2\pi i}\int_{-\infty}^{0} e^{\lambda t}\widehat{x}_+(\lambda)d\lambda - \frac{1}{2\pi i}\int_{0}^{-\infty} e^{\lambda t}\widehat{x}_-(\lambda)d\lambda.$$

Inserting $\lambda = se^{i\pi}$ into the first integral taken along the upper border and $\lambda = se^{-i\pi}$ into the second integral along the lower border, we get

$$x_3(t) = \int_0^\infty e^{-st} K_\alpha(s)\,ds.$$

Evidently, the term $x_3(t)$ vanishes as $t \to \infty$. When $\alpha \in (1,2)$ $\cos(\pi/\alpha)$ is negative and $x_2(t)$ becomes also vanishingly small at large time. Both these terms describe a relaxation process in the oscillator, and only the first term defines steady-state oscillation in this system,

$$x_{\mathrm{st}}(t) = A\sin(\omega t - \delta),$$

with driving force frequency ω, amplitude

$$A = A_{\mathrm{st}} = \frac{1}{\sqrt{\omega^{2\alpha} + \omega_0^{2\alpha} + 2\omega^\alpha \omega_0^\alpha \cos(\alpha\pi/2)}}$$

and phase shift

$$\delta = \arctan\left[\frac{\omega^\alpha \sin(\alpha\pi/2)}{\omega^2\cos(\alpha\pi/2) + \omega_0^\alpha}\right].$$

Numerical results obstained with the use of these formulas are plotted on Figs. 7.9–7.10

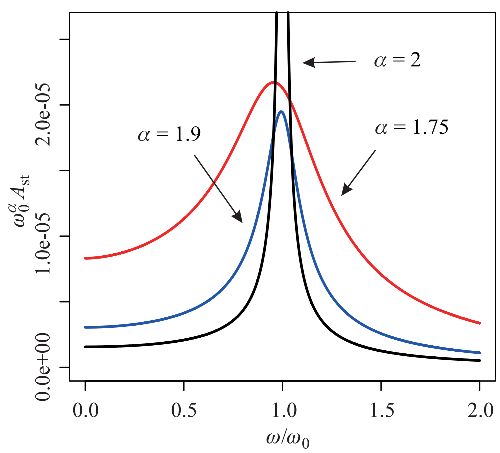

Fig. 7.9 Amplitude response for sinusoidal forcing

For other treatments of fractional oscillators see, for example, (Atanackovic and Stankovic, 2002), (Beyer and Kempfle, 1995), (Gorenflo and Mainardi, 1997), (Mainardi, 1996), (Narahari Achar et al., 2001; 2002), (Atanackovic et al., 2005) and others.

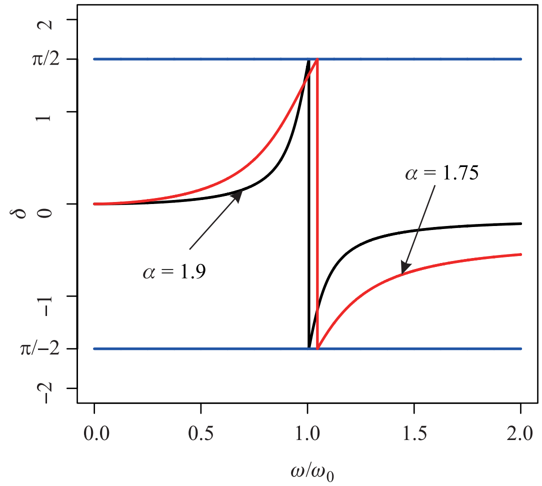

Fig. 7.10 Change in phase angle for sinusoidal forcing.

7.4.2 Linear chain of fractional oscillators

In this subsection, we consider oscillations of a one-dimensional chain of identical point masses each of them interacts with only its nearest neighbors (Narahari Achar and Hanneken, 2009).

The ordinary model of a system of masses connected by springs of spring constant k (see Fig. 7.11) is described by equation

$$m\frac{d^2u_n}{dt^2} = k(u_{n+1} - 2u_n + u_{n-1}),$$

where u_n is the displacement from the equilibrium position of the nth mass ("atom"), $2 \leqslant n \leqslant N-1$, and N is the total number of atoms. This model has been generalized to include dissipative effects by incorporating dashpots in parallel to the springs to yield the lattice dynamical version of the Kelvin–Voigt model, or the dashpots in series with the springs to yield the lattice dynamical version of the Maxwell model (Askar, 1985)

This system is easily reduced to the integral form

$$u_n(t) = u_n(0) + \dot{u}_n(0)t + \omega_0^2 \int_0^t [u_{n+1}(\tau) - 2u_n(\tau) + u_{n-1}(\tau)](t-\tau)d\tau,$$

$$2 \leqslant n \leqslant N-1,$$

where $\omega_0^2 = k/m$, in the usual notation. For the extremes, we have

$$u_1(t) = u_1(0) + \dot{u}_1(0)t + \omega_0^2 \int_0^t [u_2(\tau) - 2u_1(\tau)](t-\tau)d\tau + \int_0^t F(\tau)(t-\tau)d\tau$$

and

$$u_N(t) = u_N(0) + \dot{u}_N(0)t + \omega_0^2 \int_0^t [u_{N-1}(\tau) - 2u_N(\tau)](t-\tau)d\tau,$$

respectively. An external force $F(t)$ which applied to the end atom numbered by N is assumed to be periodic.

The integrals on the right-hand side of these equations are generalized to fractional integrals of order μ to describe the of motion of a chain of coupled fractional oscillators:

$$u_1(t) = u_1(0) + \dot{u}_1(0)t + \frac{\omega_0^\nu}{\Gamma(\nu)} \int_0^t [u_2(\tau) - 2u_1(\tau)](t-\tau)^{\nu-1}d\tau$$

$$+\frac{1}{\Gamma(1)} \int_0^t F(\tau)(t-\tau)^{\nu-1}d\tau,$$

$$u_n(t) = u_n(0) + \dot{u}_n(0)t + \frac{\omega_0^\nu}{\Gamma(\nu)} \int_0^t [u_{n+1}(\tau) - 2u_n(\tau) + u_{n-1}(\tau)](t-\tau)^{\nu-1}d\tau,$$

$$2 \leqslant n \leqslant N-1,$$

and

$$u_N(t) = u_N(0) + \dot{u}_N(0)t + \frac{\omega_0^\nu}{\Gamma(\nu)} \int_0^t [u_{N-1}(\tau) - 2u_N(\tau)](t-\tau)^{\nu-1}d\tau.$$

Taking Laplace transform of both sides of the equations yields

$$\widehat{u}_1(\lambda) = u_1(0)\lambda^{-1} + \dot{u}_1(0)\lambda^{-2} + \omega_0^\nu \lambda^{-\nu}[\widehat{u}_2(\lambda) - 2\widehat{u}_1(\lambda)] + \widehat{F}(\lambda)\lambda^{-\nu},$$

$$\widehat{u}_n(\lambda) = u_n(0)\lambda^{-1} + \dot{u}_n(0)\lambda^{-2} + \omega_0^\nu \lambda^{-\nu}[\widehat{u}_{n+1}(\lambda) - 2\widehat{u}_n(\lambda) + \widehat{u}_{n-1}(\lambda)],$$

$$2 \leqslant n \leqslant N-1,$$

and

$$\widehat{u}_N(\lambda) = u_N(0)\lambda^{-1} + \dot{u}_N(0)\lambda^{-2} + \omega_0^\nu \lambda^{-\nu}[\widehat{u}_{N-1}(\lambda) - 2\widehat{u}_N(\lambda)].$$

Then the set of linear equations can be solved numerically and the inverse Laplace transform yields the displacements as functions of time (for details see (Narahari Achar et al., 2007)).

7.4.3 *Fractionalized waves*

In the continuum limit when the number of atoms $N \to \infty$, the separation between the atoms $a \to 0$, such that the product $Na \to L$, a finite length,

$$a^2\omega_0^\nu = a^2 k/m = K/\rho,$$

where ρ is the mass density and $K = ka$ is the tension, and

$$u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)$$
$$= a^2 \frac{[u_{n+1}(t) - u_n(t)]/a - [u_n(t) - u_{n-1}(t)]/a}{a} \to a^2 \frac{\partial^2 u(x,t)}{\partial x^2}.$$

On assumption that at $t = 0$, the displacement of the free end is subject to sinusoidal forcing,

$$u(0,t) = f(t) = A\sin(\omega t),$$

we arrive at the equation

$$u(x,t) = \frac{c_0^\nu}{\Gamma(\nu)} \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2} (t-\tau)^{\nu-1} d\tau$$

with the initial conditions $u(x,0) = 0$ and $\dot{u}(0,t) = 0$ for $x > 0$ and $u(0,t) = f(t)$. Observe that c_0 has the dimension of velocity. The Laplace transform with respect to time leads to the equation

$$\frac{\partial^2 \widehat{u}(x,\lambda)}{\partial x^2} - \left(\frac{\lambda}{c_0}\right)^\nu \widehat{u}(x,\lambda) = 0$$

for $x \neq 0$ and $\widehat{u}(0,\lambda) = \widehat{f}(\lambda)$. Solving this equation,

$$\widehat{u}(x,\lambda) = \widehat{f}(\lambda) \exp\left[-(\lambda/c_0)^{\nu/2} x\right],$$

substituting

$$\widehat{f}(\lambda) = A\omega/(\omega^2 + \lambda^2)$$

and performing the inverse transformation by the Bromwich integral, we get

$$u(x,t) = \frac{A\omega}{2\pi i} \int_{\mathrm{Br}} \frac{\exp\left[\lambda t - (\lambda/c_0)^{\nu/2} x\right] d\lambda}{\omega^2 + \lambda^2} = u_{\mathrm{tr}}(x,t) + u_{\mathrm{st}}(x,t),$$

where

$$u_{\mathrm{tr}}(x,t) = \int_0^\infty ds \frac{A\omega}{\pi(s^2 + \omega^2)} \exp\left[-st - \left(\frac{s}{c_0}\right)^{\nu/2} x \cos\left(\frac{\nu\pi}{2}\right)\right] \cdot$$
$$\sin\left[\left(\frac{s}{c_0}\right)^{\nu/2} x \sin\left(\frac{\nu\pi}{2}\right)\right]$$

and

$$u_{\mathrm{st}}(x,t) = A\exp\left[-(\omega/c_0)^{\nu/2} x\cos(\nu\pi/4)\right] \sin\left[\omega t - x(\omega/c_0)^{\nu/2} \sin(\nu\pi/4)\right]$$

represent a transient part and a steady state part of the solution respectively. The first of them arises from the Hankel loop consisting of the small circle and two lines parallel to the negative x-axis and the second from the residues of the poles of the integrand. The transient part decays in time and approaches zero as $t \longrightarrow \infty$ and vanishes entirely as $\nu = 2$. Furthermore, it exhibits attenuation as a function of distance from the end as indicated by the spatial dependence of the integrand. No simple closed form expressions is obtained, the only recourse is through numerical integration (see Narahari Achar and Hanneken (2009) and references therein).

7.4.4 *Fractionalized Frenkel-Kontorova model*

We touch here upon the model suggested by Frenkel and Kontorova (1938) to describe in outline the dynamics of a crystal lattice in the vicinity of the dislocation core. The model represents a chain of atoms with the equilibrium distance a_0 (in the absence of on-site potential) interacting with the nearest neighbors via ideal springs characterized by the elastic constant k and subjected to an external harmonic force charaterized by the spatial period a_s and amplitude F_0. The equations of motion are written as

$$m\frac{d^2x_n}{dt^2} = k(x_{n+1} - 2x_n + x_{n-1}) + F_0 \sin Kx_n,$$

where the chain is supposed to be infinite with $a_0 = a_s$, when the ground state of the chain (corresponding to the minimum of the potential energy) is a commensurate structure of the atom. Treating the system as a finite chain of N atoms with periodic boundary conditions, introducing atomic displacements, and passing to the dimensionless variables, in the continuous limit we arrive at the *sine-Gordon* (SG) *equation* (for details see (Braun and Kivshar, 1998))

$$\frac{\partial^2 f}{\partial t^2} - a\frac{\partial^2 f}{\partial x^2} = b\sin(\mu f). \tag{7.27}$$

It is worth to note that the equation revealed solitonic properties of its solutions by more than a decade before the Korteweg-de Vries equation. In the sixties, the SG equation appeared in the theory of weak superconductivity and became the main non-linear equation describing the long Josephson junctions (Alfimov and Popkov, 1995; Barone and Paterno,1982), pinning in superconductors (Gurevich, 1992), motion of Bloch magnetic walls in magnetic crystals (Enz, 1964) and some other phenomena.

The physical applications of the nonlinear sine-Gordon equation are related with the description of dislocations in solid state physics (Frenkel and Kontorova,1938), motion of Bloch magnetic walls in magnetic crystals (Enz, 1964), magnetic flux propagation in superconductors (Barone and Paterno,1982). Further generalizations of this equation were performed by involving nonlocality both into the dynamic term $b\sin(\mu f)$ and into the kinetic term $a\partial^2 f/\partial x^2$ (see for review (Naumkin and Shishmarev, 1994)).

The first way has led to the equation (Cunha et al., 1996)