CHAPTER 3

Marginal Analysis for Optimal Decisions

Making optimal decisions about the levels of various business activities is an essential skill for all managers, one that requires managers to analyze benefits and costs to make the best possible decision under a given set of circumstances. When Ford Motor Company began producing its redesigned and reengineered 2002 Explorer, Ford's new CEO Jacques Nasser decided that the first 5,000 units rolling off assembly lines would not be delivered immediately to Ford dealer showrooms, even as potential buyers waited anxiously to get the new model. Instead, all of the new vehicles were parked in lots outside factories while quality control engineers examined 100 of them for defects in assembly and workmanship. The intense, 24-hour-a-day inspection process continued for three months, delaying the launch of the highly profitable new Explorer. While no one could blame Ford's executives for wanting to minimize costly product recalls—Ford was still recovering from a massive $3 billion recall of its original Explorer that had a tendency to roll over—many auto industry analysts, car buyers, and owners of Ford dealerships nonetheless thought Ford was undertaking too much quality control. Nasser assured his critics that choosing to add three months of quality control measures was optimal, or best, under the circumstances. Apparently, he believed the benefit of engaging in three months of quality control effort (the savings attributable to preventing vehicle recalls) outweighed the cost of the additional quality control measures (the loss and delay of profit as 5,000 new Explorers spent three months in factory parking lots).

As you can see, Ford's CEO, weighing costs and benefits, made a critical decision that three months, rather than two months or four months, and a sample of 100 vehicles, rather than 50 vehicles or 300 vehicles, were the optimal or best levels of these two quality control decisions for launching the redesigned Explorer. We don't have enough information about Ford's costs and benefits of quality control to tell you whether Nasser succeeded in making the optimal decision for Ford. We can, however, tell you that in April 2003, *Consumer Reports* ranked the overall reliability of Ford automobiles dead last—and Ford Motor Company had a new CEO.[1]

A manager's decision is optimal if it leads to the best outcome under a given set of circumstances. Finding the best solution involves applying the fundamental principles of optimization theory developed in this chapter. These analytical principles, which economists refer to as "marginal analysis," turn out to be nothing more than a formal presentation of commonsense ideas you already apply, probably without knowing it, in your everyday life. Marginal analysis supplies the fundamental logic for making optimal decisions. Managers benefit from understanding marginal analysis because it enables them to make better decisions while avoiding some rather common errors in business decision making.

The idea behind marginal analysis is this: When a manager contemplates whether a particular business activity needs adjusting, either more or less, to reach the best value, the manager needs to estimate how changing the activity will affect both the benefits the firm receives from engaging in the activity and the costs the firm incurs from engaging in the activity. If changing the activity level causes benefits to rise by more than costs rise, or, alternatively, costs to fall by more than benefits fall, then the net benefit the firm receives from the activity will rise. The manager should continue adjusting the activity level until no further net gains are possible, which means the activity has reached its optimal value or level.

As mentioned in Chapter 1, managers face two general types of decisions: routine business practice or tactical decisions and strategic decisions that can alter the firm's competitive environment. Marginal analysis builds the essential foundation for making everyday business decisions, such as choosing the number of workers to hire, the amount of output to produce, the amount to spend on advertising, and so on. While strategic decision making relies heavily on concepts from game theory, strategic analysis nevertheless depends indirectly on optimal decision making as the means for computing or forecasting the payoffs under various strategy options.

3.1 CONCEPTS AND TERMINOLOGY

目标函数 决策者寻找最大或最小的函数。

Optimizing behavior on the part of a decision maker involves trying to maximize or minimize an *objective function*. For a manager of a firm, the **objective function** is usually profit, which is to be maximized. For a consumer, the objective function is the satisfaction derived from consumption of goods, which is to be maximized. For

[1]See Joann Muller, "Putting the Explorer under the Microscope," *BusinessWeek*, Feb. 12, 2001, and Kathleen Kerwin, "Can Ford Pull Out of Its Skid?" *BusinessWeek*, Mar. 31, 2003.

a city manager seeking to provide adequate law enforcement services, the objective function might be cost, which is to be minimized. For the manager of the marketing division of a large corporation, the objective function is usually sales, which are to be maximized. The objective function measures whatever it is that the particular decision maker wishes to either maximize or minimize.

最大化问题 求目标函数最大化的最优化问题。

最小化问题 求目标函数最小化的最优化问题。

If the decision maker seeks to maximize an objective function, the optimization problem is called a **maximization problem.** Alternatively, if the objective function is to be minimized, the optimization problem is called a **minimization problem.** As a general rule, when the objective function measures a benefit, the decision maker seeks to maximize this benefit and is solving a maximization problem. When the objective function measures a cost, the decision maker seeks to minimize this cost and is solving a minimization problem.

自变量或选择变量 决定目标函数数值的变量。

The value of the objective function is determined by the level of one or more **activities** or **choice variables.** For example, the value of profit depends on the number of units of output produced and sold. The production of units of the good is the activity that determines the value of the objective function, which in this case is profit. The decision maker controls the value of the objective function by choosing the levels of the activities or choice variables.

离散变量 以特定间隔取值的变量。

连续变量 能在两个端点之间取任意值的变量。

The choice variables in the optimization problems discussed in this text will at times vary *discretely* and at other times vary *continuously.* A **discrete choice variable** can take on only specified integer values, such as 1, 2, 3, . . . , or 10, 20, 30, . . . Examples of discrete choice variables arise when benefit and cost data are presented in a table, where each row represents one value of the choice variable. In this text, all examples of discrete choice variables will be presented in tables. A **continuous choice variable** can take on any value between two end points. For example, a continuous variable that can vary between 0 and 10 can take on the value 2, 2.345, 7.9, 8.999, or any one of the infinite number of values between the two limits. Examples of continuous choice variables are usually presented graphically but are sometimes shown by equations. As it turns out, the optimization rules differ only slightly in the discrete and continuous cases.

无约束最优化 决策者可以不受限制地在变量中选取任何值的优化问题。

有约束最优化 决策者只能在有限的范围内选择变量值的优化问题。

In addition to being categorized as either maximization or minimization problems, optimization problems are also categorized according to whether the decision maker can choose the values of the choice variables in the objective function from an unconstrained or constrained set of values. **Unconstrained optimization** problems occur when a decision maker can choose *any* level of activity he or she wishes in order to maximize the objective function. In this chapter, we show how to solve only unconstrained *maximization* problems since all the *unconstrained* decision problems we address in this text are maximization problems. **Constrained optimization** problems involve choosing the levels of two or more activities that maximize or minimize the objective function subject to an additional requirement or constraint that restricts the values of A and B that can be chosen. An example of such a constraint arises when the total cost of the chosen activity levels must equal a specified constraint on cost. In this text, we examine both constrained maximization and constrained minimization problems.

As we show later in this chapter, the constrained maximization and the constrained minimization problems have one simple rule for the solution. Therefore, you will only have one rule to learn for all constrained optimization problems.

Even though there are a huge number of possible maximizing or minimizing decisions, you will see that all optimization problems can be solved using the single analytical technique, mentioned at the beginning of this chapter: *marginal analysis*. **Marginal analysis** involves changing the value(s) of the choice variable(s) by a small amount to see if the objective function can be further increased (in the case of maximization problems) or further decreased (in the case of minimization problems). If so, the manager continues to make incremental adjustments in the choice variables until no further improvements are possible. Marginal analysis leads to two simple rules for solving optimization problems, one for unconstrained decisions and one for constrained decisions. We turn first to the unconstrained decision.

边际分析 求解最优化问题的一个分析工具，使变量值有一个微小的变化，看它能否使得最大化问题的值进一步增大，或使得最小化问题的值进一步减小。

3.2 UNCONSTRAINED MAXIMIZATION

Any activity that decision makers might wish to undertake will generate both benefits and costs. Consequently, decision makers will want to choose the level of activity to obtain the maximum possible *net benefit* from the activity, where the **net benefit** (NB) associated with a specific amount of activity (A) is the difference between total benefit (TB) and total cost (TC) for the activity:

利润 最大化的目标函数：$NB = TB - TC$。

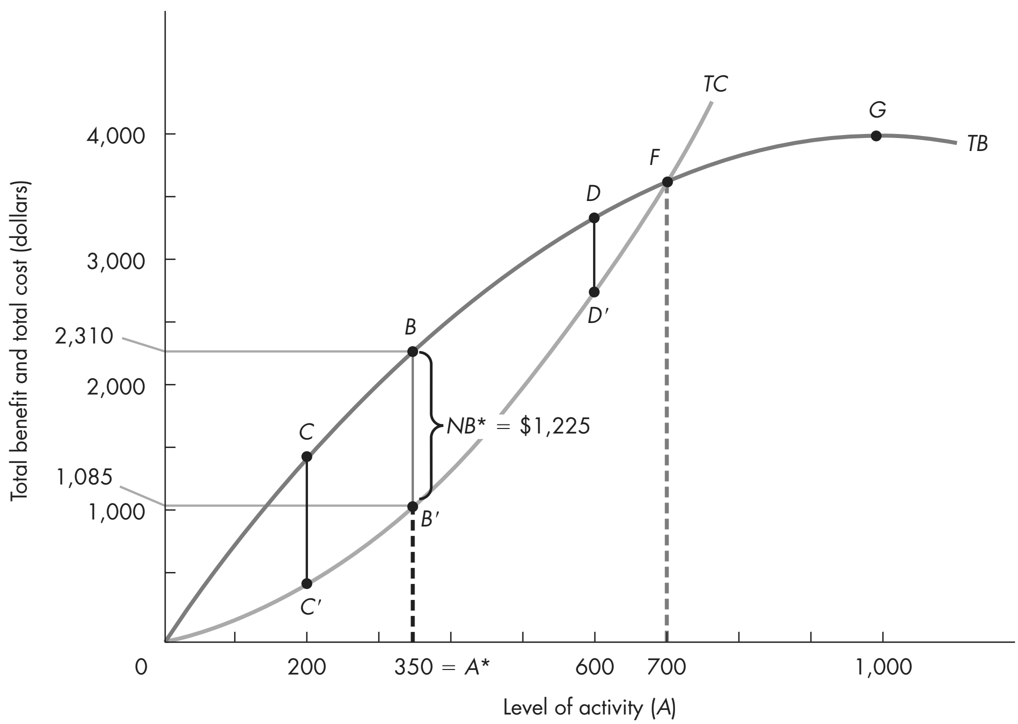
$$NB = TB - TC$$

Net benefit, then, serves as the objective function to be maximized, and the amount of activity, A, represents the choice variable. Furthermore, decision makers can choose *any* level of activity they wish, from zero to infinity, in either discrete or continuous units. Thus we are studying *unconstrained* maximization in this section.

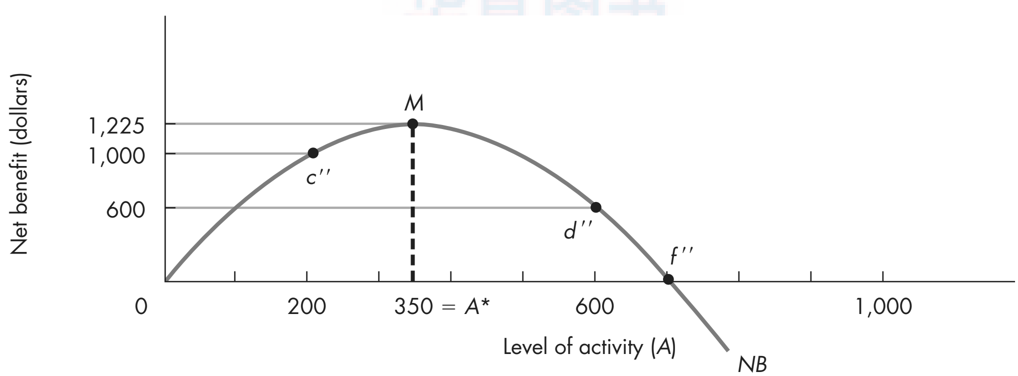
The Optimal Level of Activity (A^*)

We begin the analysis of unconstrained maximization with a rather typical set of total benefit and total cost curves for some activity, A, as shown in Panel A of Figure 3.1. Total benefit increases with higher levels of activity up to 1,000 units of activity (point G); then total benefit falls beyond this point. Total cost begins at a value of zero and rises continuously as activity increases. These "typical" curves allow us to derive general rules for finding the best solution to all such unconstrained problems, even though specific problems encountered in later chapters sometimes involve benefit and cost curves with shapes that differ somewhat from those shown in Panel A. For example, total benefit curves can be linear. Total cost curves can be linear or even S-shaped. And, as you will see in later chapters, total cost curves can include fixed costs when they take positive values at zero units of activity. In all of these variations, however, the rules for making the best decisions do not change. By learning how to solve the optimization problem as set forth in Figure 3.1, you will be prepared to solve all variations of these problems that come later in the text.

FIGURE 3.1
The Optimal Level of Activity



Panel A — Total benefit and total cost curves



Panel B — Net benefit curve

ILLUSTRATION 3.1

Is Cost–Benefit Analysis Really Useful?

We have extolled the usefulness of marginal analysis in optimal decision making—often referred to as *cost–benefit analysis*—in business decision making as well as decision making in everyday life. This process involves weighing the marginal benefits and marginal costs of an activity while ignoring all previously incurred or sunk costs. The principal rule is to increase the level of an activity if marginal benefits exceed marginal costs and decrease the level if marginal costs exceed marginal benefits. This simple rule, however, flies in the face of many honored traditional principles such as "Never give up" or "Anything worth doing is worth doing well" or "Waste not, want not." So you might wonder if cost–benefit analysis is as useful as we have said it is.

It is, at least according to an article in *The Wall Street Journal* entitled "Economic Perspective Produces Steady Yields." In this article, a University of Michigan research team concludes, "Cost–benefit analysis pays off in everyday living." This team quizzed some of the university's seniors and faculty members on such questions as how often they walk out on a bad movie, refuse to finish a bad novel, start over on a weak term paper, or abandon a research project that no longer looks promising. They believe that people who cut their losses this way are following sound economic rules: calculating the net benefits of alternative courses of action, writing off past costs that can't be recovered, and weighing the opportunity to use future time and effort more profitably elsewhere.[a]

The findings: Among faculty members, those who use cost–benefit reasoning in this fashion had higher salaries relative to their age and departments. Economists were more likely to apply the approach than professors of humanities or biology. Among students, those who have learned to use cost–benefit analysis frequently are apt to have far better grades than their SAT scores would have predicted. The more economics courses the students had taken, the more likely they were to apply cost–benefit analysis outside the classroom. The director of the University of Michigan study did concede that for many Americans cost–benefit rules often appear to conflict with traditional principles such as those we previously mentioned. Notwithstanding these probable conflicts, the study provides evidence that decision makers can indeed prosper by following the logic of marginal analysis and cost–benefit analysis.

[a]"Economic Perspective Produces Steady Yield," *The Wall Street Journal*, Mar. 31, 1992.

最优行动水平 利润最大化的产出水平（A^*）。

The level of activity that maximizes net benefit is called the **optimal level of activity,** which we distinguish from other levels of activity with an asterisk: A^*. In Panel A of Figure 3.1, net benefit at any particular level of activity is measured by the vertical distance between the total benefit and total cost curves. At 200 units of activity, for example, net benefit equals the length of line segment CC', which happens to be \$1,000 as shown in Panel B at point c''. Panel B of Figure 3.1 shows the net benefit curve associated with the *TB* and *TC* curves in Panel A. As you can see from examining the net benefit curve in Panel B, the optimal level of activity, A^*, is 350 units, where *NB* reaches its maximum value. At 350 units in Panel A, the vertical distance between *TB* and *TC* is maximized, and this maximum distance is \$1,225 (= NB^*).[2]

[2]You might, at first, have thought the optimal activity level was 700 units since two curves in Panel A of Figure 3.1 intersect at point *F,* and this situation frequently identifies "correct" answers in economics. But, as you can see in Panel B, choosing 700 units of activity creates no more net benefit than choosing to do nothing at all (i.e., choosing $A = 0$) because total benefit equals total cost at both zero and 700 units of activity.

Two important observations can now be made about A^* in unconstrained maximization problems. First, the optimal level of activity does not generally result in maximization of *total* benefits. In Panel A of Figure 3.1, you can see that total benefit is still rising at the optimal point *B*. As we will demonstrate later in this book, for one of the most important applications of this technique, profit maximization, the optimal level of production occurs at a point where revenues are not maximized. This outcome can confuse managers, especially ones who believe any decision that increases revenue should be undertaken. We will have much more to say about this later in the text. Second, the optimal level of activity in an unconstrained maximization problem does not result in minimization of total cost. In Panel A, you can easily verify that total cost isn't minimized at A^* but rather at zero units of activity.

Finding A^* in Figure 3.1 seems easy enough. A decision maker starts with the total benefit and total cost curves in Panel A and subtracts the total cost curve from the total benefit curve to construct the net benefit curve in Panel B. Then, the decision maker chooses the value of *A* corresponding to the peak of the net benefit curve. You might reasonably wonder why we are going to develop an alternative method, marginal analysis, for making optimal decisions. Perhaps the most important reason for learning how to use marginal analysis is that economists regard marginal analysis as "the central organizing principle of economic theory."[3] The graphical derivation of net benefit shown in Figure 3.1 serves only to *define* and *describe* the optimal level of activity; it does not explain *why* net benefit rises, falls, or reaches its peak. Marginal analysis, by focusing only on the changes in total benefits and total costs, provides a simple and complete explanation of the underlying forces causing net benefit to change. Understanding precisely what causes net benefit to improve makes it possible to develop simple rules for deciding when an activity needs to be increased, decreased, or left at its current level.

We are also going to show that using marginal analysis to make optimal decisions ensures that you will not consider irrelevant information about such things as fixed costs, sunk costs, or average costs in the decision-making process. As you will see shortly, decision makers using marginal analysis can reach the optimal activity level using only information about the benefits and costs *at the margin*. For this reason, marginal analysis requires less information than would be needed to construct *TB*, *TC*, and *NB* curves for all possible activity levels, as in Figure 3.1. There is no need to gather and process information for levels of activity that will never be chosen on the way to reaching A^*. For example, if the decision maker is currently at 199 units of activity in Figure 3.1, information about benefits and costs is only needed for activity levels from 200 to 351 units. The optimal level of activity can be found without any information about benefits or costs below 200 units or above 351 units.

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[3]See Robert B. Ekelund, Jr., and Robert F. Hébert, *A History of Economic Theory and Method*, 4th ed. (New York: McGraw-Hill, 1997), p. 264.

Marginal Benefit and Marginal Cost

边际收益 变量的微小增加对总收益贡献的增加量。

边际成本 变量的微小增加对总成本的增加量。

In order to understand and use marginal analysis, you must understand the two key components of this methodology: *marginal benefit* and *marginal cost*. **Marginal benefit** is the change in total benefit caused by an incremental change in the level of an activity. Similarly, **marginal cost** is the change in total cost caused by an incremental change in activity. Dictionaries typically define "incremental" to mean "a small positive or negative change in a variable." You can think of "small" or "incremental" changes in activity to be any change that is small *relative* to the total level of activity. In most applications it is convenient to interpret an incremental change as a one-unit change. In some decisions, however, it may be impractical or even impossible to make changes as small as one-unit. This causes no problem for applying marginal analysis as long as the activity can be adjusted in relatively small increments. We should also mention that "small" refers only to the change in *activity level*; "small" doesn't apply to the resulting changes in total benefit or total cost, which can be any size.

Marginal benefit and marginal cost can be expressed mathematically as

$$MB = \frac{\text{Change in total benefit}}{\text{Change in activity}} = \frac{\Delta TB}{\Delta A}$$

and

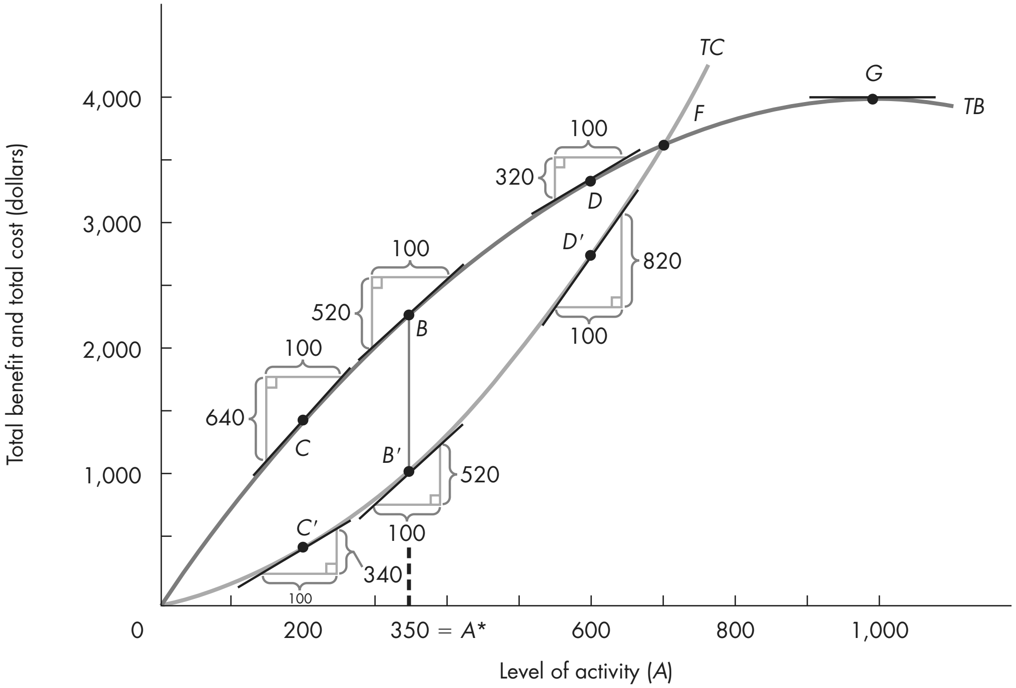
$$MC = \frac{\text{Change in total cost}}{\text{Change in activity}} = \frac{\Delta TC}{\Delta A}$$

where the symbol "Δ" means "the change in" and A denotes the level of an activity. Since "marginal" variables measure rates of change in corresponding "total" variables, marginal benefit and marginal cost are also *slopes* of total benefit and total cost curves, respectively.

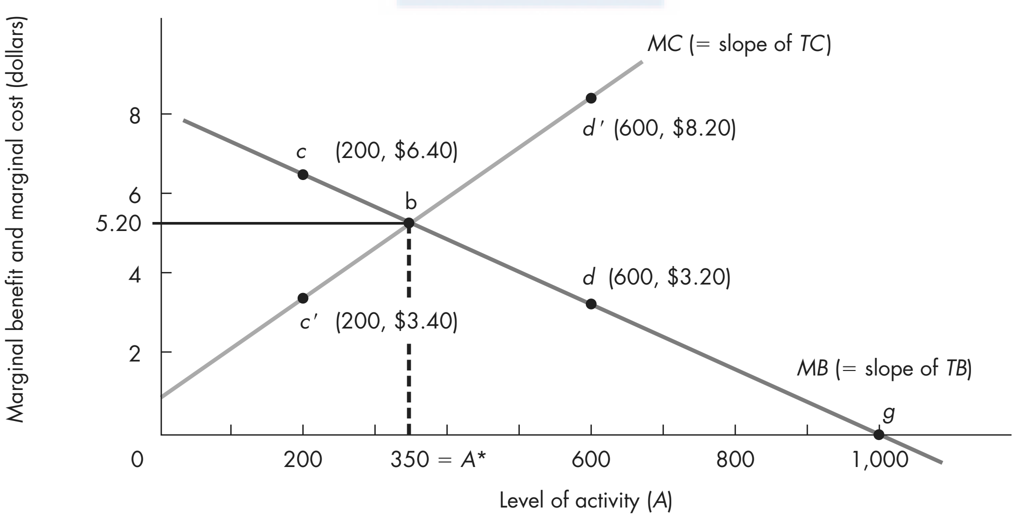
The two panels in Figure 3.2 show how the total curves in Figure 3.1 are related to their respective marginal curves. Panel A in Figure 3.2 illustrates the procedure for measuring slopes of total curves at various points or levels of activity. Recall from your high school math classes or a pre-calculus course in college that the slope of a curve at any particular point can be measured by first constructing a line tangent to the curve at the point of measure and then computing the slope of this tangent line by dividing the "rise" by the "run" of the tangent line.[4] Consider, for example, the slope of TB at point C in Panel A. The tangent line at point C rises by

[4]When a line is tangent to a curve, it touches the curve at only one point. For smooth, continuous curves, only one line can be drawn tangent to the curve at a single point. Consequently, the slope of a curve at a point is unique and equal to the slope of the tangent line at that point. The algebraic sign of the slope indicates whether the variables on the vertical and horizontal axes are directly related (a positive algebraic slope) or inversely related (a negative algebraic slope). For a concise review of measuring and interpreting slopes of curves, see "Review of Fundamental Mathematics" in the *Student Workbook* that accompanies this text.

FIGURE 3.2
Relating Marginals to Totals



Panel A — Measuring slopes along TB and TC



Panel B — Marginals give slopes of totals

640 units (dollars) over a 100-unit run, and total benefit's slope at point *C* is $6.40 (= $640/100). Thus the marginal benefit of the 200th unit of activity is $6.40, which means adding the 200th unit of activity (going from 199 to 200 units) causes total benefit to rise by $6.40.[5]

You should understand that the value of marginal benefit also tells you that subtracting the 200th unit (going from 200 to 199 units) causes total benefit to *fall* by $6.40. Since the slope of *TB* at point *C* is $6.40 per unit change in activity, marginal benefit at point *c* in Panel B is $6.40. You can verify that the same relation holds for the rest of the points shown on total benefit (*B, D,* and *G*), as well as for the points shown on total cost (*C*′, *B*′, and *D*′). We summarize this important discussion in a principle:

Principle Marginal benefit (marginal cost) is the change in total benefit (total cost) per unit change in the level of activity. The marginal benefit (marginal cost) of a particular unit of activity can be measured by the slope of the line tangent to the total benefit (total cost) curve at that point of activity.

At this point, you might be concerned that constructing tangent lines and measuring slopes of the tangent lines presents a tedious and imprecise method of finding marginal benefit and marginal cost curves. As you will see, the marginal benefit and marginal cost curves used in later chapters are obtained without drawing tangent lines. It is quite useful, nonetheless, for you to be able to visualize a series of tangent lines along total benefit and total cost curves in order to see why marginal benefit and marginal cost curves, respectively, are rising, falling, or even flat. Even if you don't know the *numerical* values of the slopes at points *C, B, D,* and *F* in Figure 3.2, you can still determine that marginal benefit in Panel B must slope downward because, as you can tell by looking, the tangent lines along *TB* get flatter (slopes get smaller) as the activity increases. Marginal cost, on the other hand, must be increasing in Panel B because, as you can tell by looking, its tangent lines get steeper (slope is getting larger) as the activity increases.

Finding Optimal Activity Levels with Marginal Analysis

As we stated earlier, the method of marginal analysis involves comparing marginal benefit and marginal cost to see if net benefit can be increased by making an incremental change in activity level. We can now demonstrate exactly how this works using the marginal benefit and marginal cost curves in Panel B of Figure 3.2. Let's suppose the decision maker is currently undertaking 199 units of activity in Panel B and wants to decide whether an incremental change in activity can

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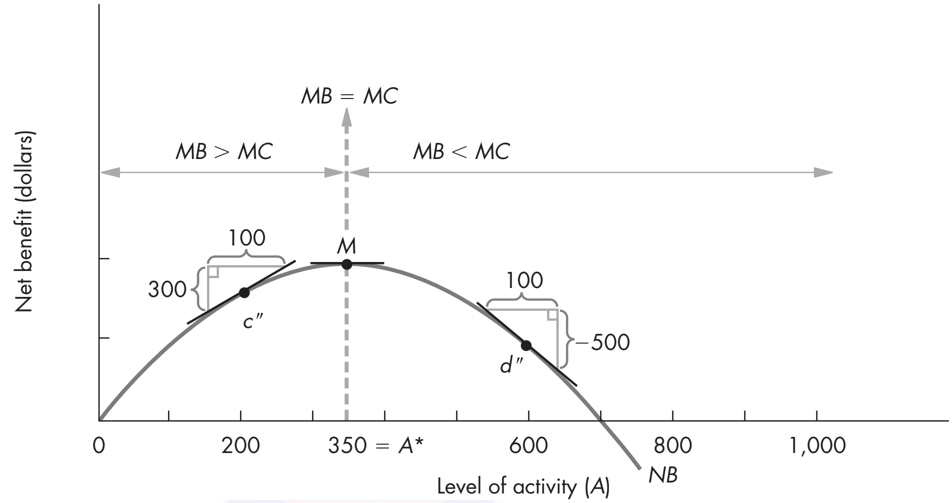
[5]When interpreting numerical values for marginal benefit and marginal cost, remember that the values refer to a particular unit of activity. In this example, marginal benefit equals $6.40 for the 200th unit. Strictly speaking, it is incorrect, and sometimes confusing, to say "marginal cost is $6.40 for 200 units." At 200 units of activity, the marginal benefit is $6.40 *for the last unit of activity undertaken* (i.e., the 200th unit).

FIGURE 3.3
Using Marginal Analysis to Find A^*



cause net benefit to rise. Adding the 200th unit of activity will cause both total benefit and total cost to rise. As you can tell from points c and c' in Panel B, TB increases by more than TC increases (\$6.40 is a larger increase than \$3.40). Consequently, increasing activity from 199 to 200 units will cause net benefit to rise by \$3 (= \$6.40 − \$3.40). Notice in Figure 3.3 that, at 200 units of activity (point c''), net benefit is rising at a rate of \$3 (= \$300/100) per unit increase in activity, as it must since MB equals \$6.40 and MC equals \$3.40.

After increasing the activity to 200 units, the decision maker then reevaluates benefits and costs at the margin to see whether another incremental increase in activity is warranted. In this situation, for the 201st unit of activity, the decision maker once again discovers that MB is greater than MC, which indicates the activity should be further increased. This incremental adjustment process continues until marginal benefit and marginal cost are exactly equal at point M ($A^* = 350$). As a practical matter, the decision maker can make a single adjustment to reach equilibrium, jumping from 199 units to 350 units in one adjustment of A, or make a series of smaller adjustments until MB equals MC at 350 units of activity. In any case, the number of adjustments made to reach A^* does not, of course, alter the optimal decision or the value of net benefit at its maximum point.

Now let's start from a position of too much activity instead of beginning with too little activity. Suppose the decision maker begins at 600 units of activity, which you can tell is too much activity by looking at the NB curve (in either Figure 3.1 or 3.3). Subtracting the 600th unit of activity will cause both total benefit and total cost to fall. As you can tell from points d and d' in Panel B of Figure 3.2, TC decreases by more than TB decreases (\$8.20 is a larger decrease than \$3.20).

TABLE 3.1
Marginal Analysis Decision Rules

	MB > *MC*	*MB* < *MC*
Increase activity	*NB* rises	*NB* falls
Decrease activity	*NB* falls	*NB* rises

Consequently, reducing activity from 600 to 599 units will cause net benefit to rise by $5 (= $8.20 − $3.20). You can now verify in Figure 3.3 that at 600 units of activity (point *d*″) net benefit is rising at a rate of $5 per unit *decrease* in activity. Since *MC* is still greater than *MB* at 599 units, the decision maker would continue reducing activity until *MB* exactly equals *MC* at 350 units (point *M*).

Table 3.1 summarizes the logic of marginal analysis by presenting the relation between marginal benefit, marginal cost, and net benefit set forth in the previous discussion and shown in Figure 3.3. We now summarize in the following principle the logic of marginal analysis for unconstrained maximization problems in which the choice variable is continuous:

Principle If, at a given level of activity, a small increase or decrease in activity causes net benefit to increase, then this level of the activity is not optimal. The activity must then be increased (if marginal benefit exceeds marginal cost) or decreased (if marginal cost exceeds marginal benefit) to reach the highest net benefit. The optimal level of the activity—the level that maximizes net benefit—is attained when no further increases in net benefit are possible for any changes in the activity, which occurs at the activity level for which marginal benefit equals marginal cost: *MB* = *MC*.

While the preceding discussion of unconstrained optimization has allowed only one activity or choice variable to influence net benefit, sometimes managers will need to choose the levels of two or more variables. As it turns out, when decision makers wish to maximize the net benefit from several activities, precisely the same principle applies: The firm maximizes net benefit when the marginal benefit from each activity equals the marginal cost of that activity. The problem is somewhat more complicated mathematically because the manager will have to equate marginal benefits and marginal costs for all of the activities *simultaneously.* For example, if the decision maker chooses the levels of two activities *A* and *B* to maximize net benefit, then the values for *A* and *B* must satisfy two conditions at once: $MB_A = MC_A$ and $MB_B = MC_B$. As it happens in this text, all the unconstrained maximization problems involve just one choice variable or activity.

T 3 4 5

Maximization with Discrete Choice Variables

In the preceding analysis, the choice variable or activity level was a continuous variable. When a choice variable can vary only discretely, the logic of marginal analysis applies in exactly the same manner as when the choice variable is continuous. However, when choice variables are discrete, decision makers will not usually

TABLE 3.2
Optimization with a Discrete Choice Variable

(1) Level of activity (*A*)	(2) Total benefit of activity (*TB*)	(3) Total cost of activity (*TC*)	(4) Net benefit of activity (*NB*)	(5) Marginal benefit (*MB*)	(6) Marginal cost (*MC*)
0	$ 0	$ 0	$ 0	—	—
1	16	2	14	16	2
2	30	6	24	14	4
3	40	11	29	10	5
4	48	20	28	8	9
5	54	30	24	6	10
6	58	45	13	4	15
7	61	61	0	3	16
8	63	80	−17	2	19

be able to adjust the level of activity to the point where marginal benefit exactly equals marginal cost. To make optimal decisions for discrete choice variables, decision makers must increase activity until the *last* level of activity is reached for which marginal benefit exceeds marginal cost. We can explain this rule for discrete choice variables by referring to Table 3.2, which shows a schedule of total benefits and total costs for various levels of some activity, *A*, expressed in integers between 0 and 8.

Let's suppose the decision maker is currently doing none of the activity and wants to decide whether to undertake the first unit of activity. The marginal benefit of the first unit of the activity is $16, and the marginal cost is $2. Undertaking the first unit of activity adds $16 to total benefit and only $2 to total cost, so net benefit rises by $14 (from $0 to $14). The decision maker would choose to undertake the first unit of activity to gain a higher net benefit. Applying this reasoning to the second and third units of activity leads again to a decision to undertake more activity. Beyond the third unit, however, marginal cost exceeds marginal benefit for additional units of activity, so no further increase beyond three units of activity will add to net benefit. As you can see, the optimal level of the activity is three units because the net benefit associated with three units ($29) is higher than for any other level of activity. These results are summarized in the following principle:

Principle When a decision maker faces an unconstrained maximization problem and must choose among discrete levels of an activity, the activity should be increased if $MB > MC$ and decreased if $MB < MC$. The optimal level of activity is reached—net benefit is maximized—when the level of activity is the last level for which marginal benefit exceeds marginal cost.

Before moving ahead, we would like to point out that this principle *cannot* be interpreted to mean "choose the activity level where *MB* and *MC* are as close to equal as possible." To see why this interpretation can lead to the wrong decision, consider the fourth unit of activity in Table 3.2. At four units of activity, *MB* (= $8)

T 6

is much closer to equality with *MC* (= \$9) than at the optimal level of activity, where *MB* (= \$10) is \$5 larger than *MC* (= \$5). Now you can see why the rule for discrete choice variables cannot be interpreted to mean "get *MB* as close to *MC* as possible."

Sunk Costs, Fixed Costs, and Average Costs Are Irrelevant

沉没成本 事前已经支付的成本。

固定成本 无论产量如何选择，都必须支付而且数量不变的成本。

In our discussion of optimization problems, we never mentioned sunk costs or fixed costs. **Sunk costs** are costs that have previously been paid and cannot be recovered. **Fixed costs** are costs that are constant and must be paid no matter what level of an activity is chosen. Such costs are totally irrelevant in decision making. They either have already been paid and cannot be recovered, as in the case of sunk costs, or must be paid no matter what a manager or any other decision maker decides to do, as in the case of fixed costs. In either case, the *only* relevant decision variables—marginal cost and marginal revenue—are in no way affected by the levels of either sunk or fixed costs.

Suppose you head your company's advertising department and you have just paid \$2 million to an advertising firm for developing and producing a 30-second television ad, which you plan to air next quarter on broadcast television networks nationwide. The \$2 million one-time payment gives your company full ownership of the 30-second ad, and your company can run the ad as many times as it wishes without making any further payments to the advertising firm for its use. Under these circumstances, the \$2 million payment is a sunk cost because it has already been paid and cannot be recovered, even if your firm decides not to use the ad after all.

To decide how many times to run the ad next quarter, you call a meeting of your company's advertising department. At the meeting, the company's media buyer informs you that 30-second television spots during *American Idol* will cost \$250,000 per spot. The marketing research experts at the meeting predict that the 24th time the ad runs it will generate \$270,000 of additional sales, while running it a 25th time will increase sales by \$210,000. Using the logic of marginal analysis, the marketing team decides running the new ad 24 times next quarter is optimal because the 24th showing of the ad is the last showing for which the marginal benefit exceeds the marginal cost of showing the ad:

$$MB = \$270{,}000 > 250{,}000 = MC$$

It would be a mistake to go beyond 24 showings, because the 25th showing would decrease net benefit; the change in net benefit would be −\$40,000 (= \$210,000 − \$250,000).

Two days after this meeting, you learn about a serious accounting error: Your company actually paid \$3 million to the advertising firm for developing and producing your company's new television ad, not \$2 million as originally reported. As you consider how to handle this new information, you realize that you don't need to call another meeting of the marketing department to reconsider its decision about running the ad 24 times next quarter. Because the amount paid to the advertising firm is a sunk cost, it doesn't affect either the marginal benefit or the marginal cost of running the ad one more time. The optimal number of times to run the ad is 24 times no matter how much the company paid in the past to obtain the ad.

Converting this example to a fixed cost, suppose that two days after your meeting you find out that, instead of making a sunk payment to buy the ad, your company instead decided to sign a 30-month contract leasing the rights to use the television ad for a monthly lease payment of $10,000. This amount is a fixed payment in each of the 30 months and must be paid no matter how many times your company decides to run the ad, even if it chooses never to run the ad. Do you need to call another meeting of the marketing department to recalculate the optimal number of times to run the ad during *American Idol?* As before, no new decision needs to be made. Since the fixed monthly loan payment does not change the predicted gain in sales (*MB*) or the extra cost of running the ad (*MC*), the optimal number of times to run the ad remains 24 times.

While you should now understand that things over which you have no control should not affect decisions, some economic experiments do, surprisingly, find that many people fail to ignore fixed or sunk costs when making decisions. They say things such as, "I've already got so much invested in this project, I have to go on with it." As you are aware, they should weigh the costs and benefits of going on before doing so. Then, if the benefits are greater than the *additional* costs, they should go on; if the *additional* costs are greater than the benefits, they should not go on. As Illustration 3.1 shows, failing to ignore fixed or sunk costs is a bad policy even in everyday decision making.

平均成本或单位成本
总成本除以行动数量得到的每单位行动水平的成本。

Another type of cost that should be ignored in finding the optimal level of an activity is the *average* or *unit cost* of the activity. **Average (or unit) cost** is the cost per unit of activity, computed by dividing total cost by the number of units of activity. In order to make optimal decisions, decision makers should not be concerned about whether their decision will push average costs up or down. The reason for ignoring average cost is quite simple: The impact on net benefit of making an incremental change in activity depends only on *marginal* benefit and *marginal* cost ($\Delta NB = MB - MC$), not on *average* benefit or *average* cost. In other words, optimal decisions are made at the margin, not "on the average."

To illustrate this point, consider the decision in Table 3.2 once again. The average cost of two units of activity is $3 (= $6/2) and average cost for three units of activity is $3.67 (= $11/3). Recall from our earlier discussion, the decision to undertake the third unit of activity is made because the marginal benefit exceeds the marginal cost ($10 > $5), and net benefit rises. It is completely irrelevant that the average cost of three units of activity is higher than the average cost of two units of activity. Alternatively, a decision maker should not decrease activity from three units to two units just to achieve a reduction in average cost from $3.67 to $3 per unit of activity; such a decision would cause net benefit to fall from $29 to $24. The following principle summarizes the role of sunk, fixed, and average costs in making optimal decisions:

T 7

Principle Decision makers wishing to maximize the net benefit of an activity should ignore any sunk costs, any fixed costs, and the average costs associated with the activity because none of these costs affect the marginal cost of the activity and so are irrelevant for making optimal decisions.

3.3 CONSTRAINED OPTIMIZATION

On many occasions a manager will face situations in which the choice of activity levels is constrained by the circumstances surrounding the maximization or minimization problem. These constrained optimization problems can be solved, as in the case of unconstrained maximization, using the logic of marginal analysis. As noted in Section 3.1, even though constrained optimization problems can be either maximization or minimization problems, the optimization rule is the same for both types.

A crucial concept for solving constrained optimization problems is the concept of marginal benefit per dollar spent on an activity. Before you can understand how to solve constrained optimization problems, you must first understand how to interpret the ratio of the marginal benefit of an activity divided by the price of the activity.

Marginal Benefit per Dollar Spent on an Activity

Retailers frequently advertise that their products give "more value for your money." People don't usually interpret this as meaning the best product in its class or the one with the highest value. Neither do they interpret it as meaning the cheapest. The advertiser wants to get across the message that customers will get more for their money or more value for each dollar spent on the product. When product rating services (such as *Consumer Reports*) rate a product a "best buy," they don't mean it is the best product or the cheapest; they mean that consumers will get more value per dollar spent on that product. When firms want to fill a position, they don't necessarily hire the person who would be the most productive in the job—that person may cost too much. Neither do they necessarily hire the person who would work for the lowest wages—that person may not be very productive. They want the employee who can do the job and give the highest productivity for the wages paid.

In these examples, phrases such as "most value for your money," "best buy," and "greatest bang per buck" mean that a particular activity yields the highest marginal benefit per dollar spent. To illustrate this concept, suppose you are the office manager for an expanding law firm and you find that you need an extra copy machine in the office—the one copier you have is being overworked. You shop around and find three brands of office copy machines (brands A, B, and C) that have virtually identical features. The three brands do differ, however, in price and in the number of copies the machines will make before they wear out. Brand A's copy machine costs $2,500 and will produce about 500,000 copies before it wears out. The marginal benefit of this machine is 500,000 ($MB_A = 500{,}000$) since the machine provides the law office with the ability to produce 500,000 additional copies.

To find the marginal benefit *per dollar spent* on copy machine *A*, marginal benefit is divided by price ($P_A = 2{,}500$):

$$\begin{aligned} MB_A/P_A &= 500{,}000 \text{ copies}/2{,}500 \text{ dollars} \\ &= 200 \text{ copies}/\text{dollar} \end{aligned}$$

You get 200 copies for each of the dollars spent to purchase copy machine *A*.

Now compare machine *A* with machine *B*, which will produce 600,000 copies and costs $4,000. The marginal benefit is greater, but so is the price. To determine how "good a deal" you get with machine *B*, compute the marginal benefit per dollar spent on machine *B:*

$$\begin{aligned} MB_B/P_B &= 600{,}000 \text{ copies}/4{,}000 \text{ dollars} \\ &= 150 \text{ copies}/\text{dollar} \end{aligned}$$

Even though machine *B* provides a higher marginal benefit, its marginal benefit per dollar spent is lower than that for machine *A*. Machine *A* is a better deal than machine *B* because it yields higher marginal benefit per dollar. The third copy machine produces 580,000 copies over its useful life and costs $2,600. Machine *C* is neither the best machine (580,000 < 600,000 copies) nor is it the cheapest machine ($2,600 > $2,500), but of the three machines, machine *C* provides the greatest marginal benefit per dollar spent:

$$\begin{aligned} MB_C/P_C &= 580{,}000 \text{ copies}/2{,}600 \text{ dollars} \\ &= 223 \text{ copies}/\text{dollar} \end{aligned}$$

On a bang per buck basis, you would rank machine *C* first, machine *A* second, and machine *B* third.

When choosing among different activities, a decision maker compares the marginal benefits per dollar spent on each of the activities. Marginal benefit (the "bang"), *by itself*, does not provide sufficient information for decision-making purposes. Price (the "buck"), *by itself*, does not provide sufficient information for making decisions. It is marginal benefit per dollar spent (the "bang per buck") that matters in decision making.

T 8

Constrained Maximization

In the general constrained maximization problem, a manager must choose the levels of two or more activities in order to maximize a total benefit (objective) function subject to a constraint in the form of a budget that restricts the amount that can be spent.[6] Consider a situation in which there are two activities, *A* and *B*.

[6]It may look like constrained maximization problems no longer use net benefit as the objective function to be maximized. Note, however, that maximizing total benefit while total cost must remain constant in order to meet a budget constraint does indeed result in the maximum possible amount of net benefit for a given level of total cost.

Each unit of activity A costs \$4 to undertake, and each unit of activity B costs \$2 to undertake. The manager faces a constraint that allows a total expenditure of only \$100 on activities A and B combined. The manager wishes to allocate \$100 between activities A and B so that the combined total benefit from both activities is maximized.

The manager is currently choosing 20 units of activity A and 10 units of activity B. The constraint is met for the combination 20A and 10B since $(\$4 \times 20) + (\$2 \times 10) = \$100$. For this combination of activities, suppose that the marginal benefit of the last unit of activity A is 40 units of additional benefit and the marginal benefit of the last unit of B is 10 units of additional benefit. In this situation, the marginal benefit per dollar spent on activity A exceeds the marginal benefit per dollar spent on activity B:

$$\frac{MB_A}{P_A} = \frac{40}{4} = 10 > 5 = \frac{10}{2} = \frac{MB_B}{P_B}$$

Spending an additional dollar on activity A increases total benefit by 10 units, while spending an additional dollar on activity B increases total benefit by 5 units. Since the marginal benefit per dollar spent is greater for activity A, it provides "more bang per buck" or is a better deal at this combination of activities.

To take advantage of this fact, the manager can increase activity A by one unit and decrease activity B by two units (now, $A = 21$ and $B = 8$). This combination of activities still costs \$100 $[(\$4 \times 21) + (\$2 \times 8) = \$100]$. Purchasing one more unit of activity A causes total benefit to rise by 40 units, while purchasing two less units of activity B causes total benefit to fall by 20 units. The combined total benefit from activities A and B *rises* by 20 units $(= 40 - 20)$ *and* the new combination of activities ($A = 21$ and $B = 8$) costs the same amount, \$100, as the old combination. Total benefit riser without spending any more than \$100 on the activities.

Naturally, the manager will continue to increase spending on activity A and reduce spending on activity B as long as MB_A/P_A exceeds MB_B/P_B. In most situations, the marginal benefit of an activity declines as the activity increases.[7] Consequently, as activity A is increased, MB_A gets smaller. As activity B is decreased, MB_B gets larger. Thus as spending on A rises and spending on B falls, MB_A/P_A falls and MB_B/P_B rises. A point is eventually reached at which activity A is no longer a better deal than activity B; that is, MB_A/P_A equals MB_B/P_B. At this point, total benefit is maximized subject to the constraint that only \$100 is spent on the two activities.

If the original allocation of spending on activities A and B had been one where

$$\frac{MB_A}{P_A} < \frac{MB_B}{P_B}$$

[7]Decreasing marginal benefit is quite common. As you drink several cans of Coke in succession, you get ever smaller amounts of additional satisfaction from successive cans. As you continue studying for an exam, each additional hour of study increases your expected exam grade by ever smaller amounts. In such cases, marginal benefit is inversely related to the level of the activity. Increasing the activity causes marginal benefit to fall, and decreasing the activity level causes marginal benefit to rise.

the manager would recognize that activity B is the better deal. In this case, total benefit could be increased by spending more on activity B and less on activity A while maintaining the $100 budget. Activity B would be increased by two units for every one-unit decrease in activity A (in order to satisfy the $100 spending constraint) until the marginal benefit per dollar spent is equal for both activities:

$$\frac{MB_A}{P_A} = \frac{MB_B}{P_B}$$

If there are more than two activities in the objective function, the condition is expanded to require that the marginal benefit per dollar spent be equal for all activities.

Principle To maximize total benefits subject to a constraint on the levels of activities, choose the level of each activity so that the marginal benefit per dollar spent is equal for all activities

$$\frac{MB_A}{P_A} = \frac{MB_B}{P_B} = \frac{MB_C}{P_C} = \cdots = \frac{MB_Z}{P_Z}$$

T 9 10

and at the same time, the chosen level of activities must also satisfy the constraint.

Optimal Advertising Expenditures: An Example of Constrained Maximization

To illustrate how a firm can use the technique of constrained maximization to allocate its advertising budget, suppose a manager of a small retail firm wants to maximize the effectiveness (in total sales) of the firm's weekly advertising budget of $2,000. The manager has the option of advertising on the local television station or on the local AM radio station. As a class project, a marketing class at a nearby college estimated the impact on the retailer's sales of varying levels of advertising in the two different media. The manager wants to maximize the number of units sold; thus the total benefit is measured by the total number of units sold. The estimates of the *increases* in weekly sales (the marginal benefits) from increasing the levels of advertising on television and radio are given in columns 2 and 4 below:

(1) Number of ads	(2) MB_{TV}	(3) $\frac{MB_{TV}}{P_{TV}}$	(4) MB_{radio}	(5) $\frac{MB_{radio}}{P_{radio}}$
1	400	1.0	360	1.2
2	300	0.75	270	0.9
3	280	0.7	240	0.8
4	260	0.65	225	0.75
5	240	0.6	150	0.5
6	200	0.5	120	0.4

Television ads are more "powerful" than radio ads in the sense that the marginal benefits from additional TV ads tend to be larger than those for more radio ads. However, since the manager is constrained by the limited advertising budget, the relevant measure is not simply marginal benefit but, rather, marginal benefit per dollar spent on advertising. The price of television ads is $400 per ad, and the price of radio ads is $300 per ad. Although the first TV ad dominates the first radio ad in terms of its marginal benefit (increased sales), the marginal benefit per dollar's worth of expenditure for the first radio ad is greater than that for the first television ad:

	Marginal benefit/price	
	Television	Radio
Ad 1	400/400 = 1.00	360/300 = 1.2

This indicates that sales rise by 1 unit per dollar spent on the first television ad and 1.2 units on the first radio ad. Therefore, when the manager is allocating the budget, the first ad she selects will be a radio ad—the activity with the larger marginal benefit per dollar spent. Following the same rule and using the MB/P values in columns 3 and 5 above, the $2,000 advertising budget would be allocated as follows:

Decision	*MB/P*	Ranking of *MB/P*	Cumulative expenditures
Buy radio ad 1	360/300 = 1.20	1	$ 300
Buy TV ad 1	400/400 = 1.00	2	700
Buy radio ad 2	270/300 = 0.90	3	1,000
Buy radio ad 3	240/300 = 0.80	4	1,300
Buy TV ad 2	300/400 = 0.75		1,700
Buy radio ad 4	225/300 = 0.75	5 (tie)	2,000

By selecting two television ads and four radio ads, the manager of the firm has maximized sales subject to the constraint that only $2,000 can be spent on advertising activity. Note that for the optimal levels of television and radio ads (two TV and four radio):

$$\frac{MB_{\text{TV}}}{P_{\text{TV}}} = \frac{MB_{\text{radio}}}{P_{\text{radio}}} = 0.75$$

T 11 12

The fact that the preceding application used artificially simplistic numbers shouldn't make you think that the problem is artificial. If we add a few zeros to the prices of TV and radio ads, we have the real-world situation faced by advertisers.

Constrained Minimization

Constrained minimization problems involve minimizing a total cost function (the objective function) subject to a constraint that the levels of activities be chosen

ILLUSTRATION 3.2

Seattle Seahawks Win on "Bang Per Buck" Defense

Behind every professional sports team, a team of business decision makers is constantly at work—there is no off-season for the business team—trying to figure out how to put together the most profitable team of players. In the NFL, the team-building process is a *constrained* optimization problem because the football league imposes restrictions on the amount each team can spend on players in a season, as well as the number of players the team can carry on its roster. Currently, NFL teams are limited to 53 players and a salary cap of $85 million per season. While teams can, and do, structure cash bonuses to players in ways that allow them to exceed the salary caps in any single year, the NFL spending constraint nonetheless restricts the total amount a team can spend on its players. Based on what you have learned in this chapter about constrained optimization, it should come as no surprise to you that, in the business of sports, finding and keeping players who can deliver the greatest bang for the buck may be the most important game a team must play. History has shown that most teams making it to the Super Bowl have played the "bang for the buck" game very well.

A recent article in *The Wall Street Journal*[a] illustrates how much importance is attached to the "game behind the game":

> Every year, from the moment the last piece of Super Bowl confetti is bagged up, the NFL's army of bean counters and personnel directors starts to deconstruct the strategy of the winning team to see what lessons they can learn. But they don't start with the game films. The first order of business is to check the team's salary-cap figures.

To see how personnel directors of NFL teams follow the principles of constrained maximization in choosing their teams' rosters, we can look at the story of the Seattle Seahawks, who played (and lost to) the Pittsburgh Steelers in Super Bowl XL. According to the *WSJ* article, the Seahawks' personnel director, Tim Ruskell, who enjoyed wide acclaim for building the highly regarded defenses at Tampa Bay and Atlanta, faced a particularly harsh salary-cap constraint in Seattle for the 2005 football season. The Seahawks' salary cap in 2005 was penalized by $18 million of "dead money"—the term used for money paid by previous contract to players no longer on the team—so Ruskell began with only $67 million to spend on players. Making matters even worse for the team's chief business decision maker was the fact that Seattle had signed giant contracts, even by NFL standards, to

such that a given level of total benefit is achieved. Consider a manager who must minimize the total cost of two activities, A and B, subject to the constraint that 3,000 units of benefit are to be generated by those activities. The price of activity A is $5 per unit and the price of activity B is $20 per unit. Suppose the manager is currently using 100 units of activity A and 60 units of activity B and this combination of activity generates total benefit equal to 3,000. At this combination of activities, the marginal benefit of the last unit of activity A is 30 and the marginal benefit of the last unit of activity B is 60. In this situation, the marginal benefit per dollar spent on activity A exceeds the marginal benefit per dollar spent on activity B:

$$\frac{MB_A}{P_A} = \frac{30}{5} = 6 > 3 = \frac{60}{20} = \frac{MB_B}{P_B}$$

Since the marginal benefit per dollar spent is greater for activity A than for activity B, activity A gives "more for the money."

keep its biggest stars on offense. Obviously, this left Ruskell with very little money to spend on building the Seahawks' defense. Compared with its Super Bowl rival, Seattle spent $11 million less on defense than did the Steelers. Ruskell's strategy for hiring defensive players, then, had to be extremely effective if Seattle was to have any chance of going to the Super Bowl in 2005.

The way that Ruskell built Seattle's defense, subject to a very tight spending constraint, drew high praise from others in the league: "They did it (built a defense) without breaking the bank at Monte Carlo, and I think that's extremely impressive," remarked Gil Brandt, a former personnel director for the Dallas Cowboys.[b] As you know from our discussion in this chapter, Ruskell must have been very successful at finding defensive players who could deliver the highest possible marginal benefits per dollar spent. To accomplish this, he recruited only inexpensive draft picks and young free agents, who were also likely to play with "exuberance" and perform defensive tasks well enough to get to the Super Bowl. We must stress that Ruskell's strategy depended crucially on *both* the numerator and denominator in the MB/MC ratio. He understood that simply hiring the cheapest players would not produce a Super Bowl team. Team scouts had to find players who would also deliver high marginal benefits to the team's defensive squad by making lots of tackles and intercepting lots of passes. Perhaps the best example of Ruskell's success at getting the most "bang for the buck" in 2005 was Lofa Tatapu, a rookie linebacker. Tatapu, who was thought by many team scouts to be too small to be a great linebacker, became a star defensive player for Seattle and cost the team only $230,000—one-tenth the amount paid on average for linebackers in the NFL.

As you can see from this Illustration, making optimal constrained maximization decisions in practice takes not only skill and experience, it sometimes involves a bit of luck! History shows, however, that NFL personnel directors who spend their salary caps to get either the very best players (the greatest bang) or the very cheapest players (the smallest buck), don't usually make it to the Super Bowl. Thus, on Super Bowl game day, fans can generally expect to see the two NFL teams with the highest overall MB/MC ratios. Of course *winning* the Super Bowl is just a betting matter.

[a]This Illustration is based on Sam Walker, "Holding the Line on Defense Spending: Does It Pay to Hire on the Cheap?" *The Wall Street Journal*, Online Edition, Feb. 3, 2006.

[b]As quoted in Walker, "Holding the Line."

To take advantage of activity A, the manager can reduce activity B by one unit, causing total benefit to fall by 60 units and reducing cost by $20. To hold total benefit constant, the 60 units of lost benefit can be made up by increasing activity A by two units with a marginal benefit of 30 each. The two additional units of activity A cause total cost to rise by $10. By reducing activity B by one unit and increasing activity A by two units, the manager *reduces* total cost by $10 (= $20 − $10) without reducing total benefit.

As long as $MB_A/P_A > MB_B/P_B$, the manager will continue to increase activity A and decrease activity B at the rate that holds TB constant until

$$\frac{MB_A}{P_A} = \frac{MB_B}{P_B}$$

If there are more than two activities in the objective function, the condition is expanded to require that the marginal benefit per dollar spent be equal for all activities.

Principle In order to minimize total costs subject to a constraint on the levels of activities, choose the level of each activity so that the marginal benefit per dollar spent is equal for all activities

$$\frac{MB_A}{P_A} = \frac{MB_B}{P_B} = \frac{MB_C}{P_C} = \cdots = \frac{MB_Z}{P_Z}$$

and at the same time, the chosen level of activities must also satisfy the constraint.

T 13 As you can see, this is the same condition that must be met in the case of constrained maximization.

3.4 SUMMARY

In this chapter we have given you the key to the kingdom of economic decision making: marginal analysis. Virtually all of microeconomics involves solutions to optimization problems. The most interesting and challenging problems facing a manager involve trying either to maximize or to minimize particular objective functions. Regardless of whether the optimization involves maximization or minimization, or constrained or unconstrained choice variables, all optimization problems are solved by using marginal analysis.

The results of this chapter fall neatly into two categories: the solution to unconstrained and the solution to constrained optimization problems. When the values of the choice variables are *not* restricted, the optimization problem is said to be unconstrained. Unconstrained maximization problems can be solved by following this simple rule: To maximize net benefit, increase or decrease the level of activity until the marginal benefit from the activity equals the marginal cost of the activity:

$$MB = MC$$

When the choice variable is not continuous but discrete, it may not be possible to precisely equate MB and MC. For discrete choice variables, the decision maker simply carries out the activity up to the point where *any further* increases in the activity result in marginal cost exceeding marginal benefit.

Marginal analysis shows why decision makers should ignore average costs, fixed costs, and sunk costs when making decisions about the optimal level of activities. Since it is *marginal* cost that must equal marginal benefit to reach the optimal level of activity, any other cost is irrelevant for making decisions about how much of an activity to undertake.

In many instances, managers face limitations on the range of values that the choice variables can take. To maximize or minimize an objective function subject to a constraint, the ratios of the marginal benefit to price must be equal for all activities,

$$\frac{MB_A}{P_A} = \frac{MB_B}{P_B} = \cdots = \frac{MB_Z}{P_Z}$$

and the values of the choice variables must meet the constraint.

The two decision rules presented in this chapter will be used throughout this text. These rules, although simple, are the essential tools for making economic decisions.

TECHNICAL PROBLEMS

1. For each of the following decision-making problems, determine whether the problem involves constrained or unconstrained optimization; what the objective function is and, for each constrained problem, what the constraint is; and what the choice variables are.

a. We have received a foundation grant to purchase new PCs for the staff. You decide what PCs to buy.

b. We aren't earning enough profits. Your job is to redesign our advertising program and decide how much TV, direct-mail, and magazine advertising to use. Whatever we are doing now isn't working very well.

c. We have to meet a production quota but think we are going to spend too much doing so. Your job is to reallocate the machinery, the number of workers, and the raw materials needed to meet the quota.

2. Refer to Figure 3.2 on page 94 and answer the following questions:

a. At 600 units of activity, marginal benefit is ________ (rising, constant, positive, negative) because the tangent line at *D* is sloping ________ (downward, upward).

b. The marginal benefit of the 600th unit of activity is $________. Explain how this value of marginal benefit can be computed.

c. At 600 units of activity, decreasing the activity by one unit causes total benefit to ________ (increase, decrease) by $________. At point *D*, total benefit changes at a rate ________ times as much as activity changes, and *TB* and *A* are moving in the ________(same, opposite) direction, which means *TB* and *A* are ________ (directly, inversely) related.

d. At 1,000 units of activity, marginal benefit is ________. Why?

e. The marginal cost of the 600th unit of activity is $________. Explain how this value of marginal cost can be computed.

f. At 600 units of activity, decreasing the activity by one unit causes total cost to ________(increase, decrease) by $________. At point *D*′, total cost changes at a rate ________ times as much as activity changes, and *TC* and *A* are moving in the ________ (same, opposite) direction, which means *TC* and *A* are ________ (directly, inversely) related.

g. Visually, the tangent line at point *D* appears to be ________ (flatter, steeper) than the tangent line at point *D*′, which means that ________ (*NB*, *TB*, *TC*, *MB*, *MC*) is larger than ________ (*NB*, *TB*, *TC*, *MB*, *MC*).

h. Since point *D* lies above point *D*′, ________ (*NB*, *TB*, *TC*, *MB*, *MC*) is larger than ________ (*NB*, *TB*, *TC*, *MB*, *MC*), which means that ________ (*NB*, *TB*, *TC*, *MB*, *MC*) is ________ (rising, falling, constant, positive, negative, zero).

3. Fill in the blanks below. In an unconstrained maximization problem:

a. An activity should be increased if ________ exceeds ________.

b. An activity should be decreased if ________ exceeds ________.

c. The optimal level of activity occurs at the activity level for which ________ equals ________.

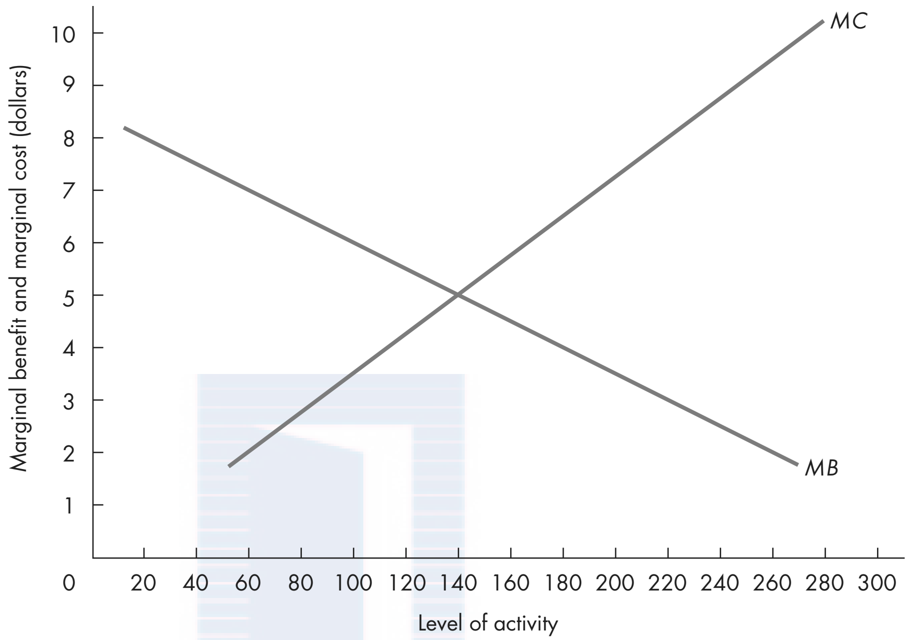
d. At the optimal level of activity, ________ is maximized, and the slope of ________ equals the slope of ________.

e. If total cost is falling faster than total benefit is falling, the activity should be ________.

f. If total benefit is rising at the same rate that total cost is rising, the decision maker should ______________.

g. If net benefit is rising, then total benefit must be rising at a rate ________ (greater than, less than, equal to) the rate at which total cost is ________ (rising, falling).

4. Use the graph below to answer the following questions:



a. At 60 units of the activity, marginal benefit is $________ and marginal cost is $________.

b. Adding the 60th unit of the activity causes net benefit to ________ (increase, decrease) by $________.

c. At 220 units of the activity, marginal benefit is $________ and marginal cost is $________.

d. Subtracting the 220th unit of the activity causes net benefit to ________ (increase, decrease) by $________.

e. The optimal level of the activity is ________________ units. At the optimal level of the activity, marginal benefit is $________ and marginal cost is $________.

5. Fill in the blanks in the following statement:

If marginal benefit exceeds marginal cost, then increasing the level of activity by one unit ________ (increases, decreases) ________ (total, marginal, net) benefit by more than it ________ (increases, decreases) ________ (total, marginal) cost. Therefore, ________ (increasing, decreasing) the level of activity by one unit must increase net benefit. The manager should continue to ________ (increase, decrease) the level of activity until marginal benefit and marginal cost are ________ (zero, equal).

6. Fill in the blanks in the following table to answer the questions below.

A	*TB*	*TC*	*NB*	*MB*	*MC*
0	$ 0	$___	$ 0		
1	___	___	27	$35	$___
2	65	___	___	___	10
3	85	30	___	___	___
4	___	___	51	___	14
5	___	60	___	8	___
6	___	___	___	5	20

a. What is the optimal level of activity in the table above?

b. What is the value of net benefit at the optimal level of activity? Can net benefit be increased by moving to any other level of *A*? Explain.

c. Using the numerical values in the table, comment on the statement, "The optimal level of activity occurs where marginal benefit is closest to marginal cost."

7. Now suppose the decision maker in Technical Problem 6 faces a fixed cost of $24. Fill in the blanks in the following table to answer the questions below. *AC* is the average cost per unit of activity.

A	*TB*	*TC*	*NB*	*MB*	*MC*	*AC*
0	$ 0	___	−$24			
1	___	___	3	$35	$___	$32
2	65	___	___	___	10	___
3	85	54	___	___	___	___
4	___	___	27	___	14	___
5	___	___	___	8	___	16.80
6	___	___	___	5	20	___

a. How does adding $24 of fixed costs affect total cost? Net benefit?

b. How does adding $24 of fixed cost affect marginal cost?

c. Compared to *A** in Technical Problem 6, does adding $24 of fixed cost change the optimal level of activity? Why or why not?

d. What advice can you give decision makers about the role of fixed costs in finding *A**?

e. What level of activity minimizes average cost per unit of activity? Is this level also the optimal level of activity? Should it be? Explain.

f. Suppose a government agency requires payment of a one-time, nonrefundable license fee of $100 to engage in activity *A*, and this license fee was paid last month. What kind of cost is this? How does this cost affect the decision maker's choice of activity level now? Explain.

8. You are interviewing three people for one sales job. On the basis of your experience and insight, you believe Jane can sell 600 units a day, Joe can sell 450 units a day, and Joan can sell 400 units a day. The daily salary each person is asking is as follows: Jane, $200; Joe, $150; and Joan, $100. How would you rank the three applicants?

9. Fill in the blanks. When choosing the levels of two activities, *A* and *B*, in order to maximize total benefits within a given budget:
   a. If at the given levels of *A* and *B*, *MB*/*P* of *A* is ________ *MB*/*P* of *B*, increasing *A* and decreasing *B* while holding expenditure constant will increase total benefits.
   b. If at the given levels of *A* and *B*, *MB*/*P* of *A* is ________ *MB*/*P* of *B*, increasing *B* and decreasing *A* while holding expenditure constant will increase total benefits.
   c. The optimal levels of *A* and *B* are the levels at which ________ equals ________.
10. A decision maker is choosing the levels of two activities, *A* and *B*, so as to maximize total benefits under a given budget. The prices and marginal benefits of the last units of *A* and *B* are denoted P_A, P_B, MB_A, and MB_B.
   a. If $P_A = \$20$, $P_B = \$15$, $MB_A = 400$, and $MB_B = 600$, what should the decision maker do?
   b. If $P_A = \$20$, $P_B = \$30$, $MB_A = 200$, and $MB_B = 300$, what should the decision maker do?
   c. If $P_A = \$20$, $P_B = \$40$, $MB_A = 300$, and $MB_B = 400$, how many units of *A* can be obtained if *B* is reduced by one unit? How much will benefits increase if this exchange is made?
   d. If the substitution in part *c* continues to equilibrium and MB_A falls to 250, what will MB_B be?
11. A decision maker wishes to maximize the total benefit associated with three activities, *X*, *Y*, and *Z*. The price per unit of activities *X*, *Y*, and *Z* is \$1, \$2, and \$3, respectively. The following table gives the ratio of the marginal benefit to the price of the activities for various levels of each activity:

Level of activity	$\frac{MB_X}{P_X}$	$\frac{MB_Y}{P_Y}$	$\frac{MB_Z}{P_Z}$
1	10	22	14
2	9	18	12
3	8	12	10
4	7	10	9
5	6	6	8
6	5	4	6
7	4	2	4
8	3	1	2

   a. If the decision maker chooses to use one unit of *X*, one unit of *Y*, and one unit of *Z*, the total benefit that results is \$________.
   b. For the fourth unit of activity *Y*, each dollar spent increases total benefit by \$________. The fourth unit of activity *Y* increases total benefit by \$________.
   c. Suppose the decision maker can spend a total of only \$18 on the three activities. What is the optimal level of *X*, *Y*, and *Z*? Why is this combination optimal? Why is the combination 2*X*, 2*Y*, and 4*Z* not optimal?
   d. Now suppose the decision maker has \$33 to spend on the three activities. What is the optimal level of *X*, *Y*, and *Z*? If the decision maker has \$35 to spend, what is the optimal combination? Explain.
12. Suppose a firm is considering two different activities, *X* and *Y*, which yield the total benefits presented in the schedule below. The price of *X* is \$2 per unit, and the price of *Y* is \$10 per unit.

Level of activity	Total benefit of activity X (TB_X)	Total benefit of activity Y (TB_Y)
0	$ 0	$ 0
1	30	100
2	54	190
3	72	270
4	84	340
5	92	400
6	98	450

a. The firm places a budget constraint of $26 on expenditures on activities *X* and *Y*. What are the levels of *X* and *Y* that maximize total benefit subject to the budget constraint?

b. What is the total benefit associated with the optimal levels of *X* and *Y* in part *a*?

c. Now let the budget constraint increase to $58. What are the optimal levels of *X* and *Y* now? What is the total benefit when the budget constraint is $58?

13. *a.* If, in a constrained minimization problem, $P_A = \$10$, $P_B = \$10$, $MB_A = 600$, and $MB_B = 300$ and one unit of *B* is taken away, how many units of *A* must be added to keep benefits constant?

b. If the substitution in part *a* continues to equilibrium, what will be the equilibrium relation between MB_A and MB_B?

APPLIED PROBLEMS

1. Using optimization theory, analyze the following quotations:

a. "The optimal number of traffic deaths in the United States is zero."

b. "Any pollution is too much pollution."

c. "We cannot pull U.S. troops out of Iraq. We have committed so much already."

d. "If Congress cuts out the NASA space station, we will have wasted all the resources that we have already spent on it. Therefore, we must continue funding it."

e. "Since JetGreen Airways has experienced a 25 percent increase in its insurance premiums, the airline should increase the number of passengers it serves next quarter in order to spread the increase in premiums over a larger number of tickets."

2. Appalachian Coal Mining believes that it can increase labor productivity and, therefore, net revenue by reducing air pollution in its mines. It estimates that the marginal cost function for reducing pollution by installing additional capital equipment is

$$MC = 40P$$

where *P* represents a reduction of one unit of pollution in the mines. It also feels that for every unit of pollution reduction the marginal increase in revenue (*MR*) is

$$MR = 1{,}000 - 10P$$

How much pollution reduction should Appalachian Coal Mining undertake?

3. Two partners who own Progressive Business Solutions, which currently operates out of an office in a small town near Boston, just discovered a vacancy in an office building in downtown Boston. One of the partners favors moving downtown because she believes the additional business gained by moving downtown will exceed the higher rent at the downtown location plus the cost of making the move. The other partner at PBS opposes moving downtown. He argues, "We have already paid for office stationery, business cards, and a large sign that cannot be moved or sold. We have spent so much on our current office that we can't afford to waste this money by moving now." Evaluate the second partner's advice not to move downtown.
4. Twentyfirst Century Electronics has discovered a theft problem at its warehouse and has decided to hire security guards. The firm wants to hire the optimal number of security guards. The following table shows how the number of security guards affects the number of radios stolen per week.

Number of security guards	Number of radios stolen per week
0	50
1	30
2	20
3	14
4	8
5	6

   a. If each security guard is paid $200 a week and the cost of a stolen radio is $25, how many security guards should the firm hire?
   b. If the cost of a stolen radio is $25, what is the most the firm would be willing to pay to hire the first security guard?
   c. If each security guard is paid $200 a week and the cost of a stolen radio is $50, how many security guards should the firm hire?
5. U.S. Supreme Court Justice Stephen Breyer's book *Breaking the Vicious Circle: Toward Effective Risk Regulation* (1993) examines government's role in controlling and managing the health risks society faces from exposure to environmental pollution. One major problem examined in the book is the cleanup of hazardous waste sites. Justice Breyer was extremely critical of policymakers who wish to see waste sites 100 percent clean.
   a. Explain, using the theory of optimization and a graph, the circumstances under which a waste site could be made "too clean." (Good answers are dispassionate and employ economic analysis.)
   b. Justice Breyer believes that society can enjoy virtually all the health benefits of cleaning up a waste site for only a "small fraction" of the total cost of completely cleaning a site. Using graphical analysis, illustrate this situation. (*Hint:* Draw *MB* and *MC* curves with shapes that specifically illustrate this situation.)
6. In Illustration 3.1 we noted that the rule for maximization set forth in the text contradicts some honored traditional principles such as "Never give up," "Anything worth doing is worth doing well," or "Waste not, want not." Explain the contradiction for each of these rules.
7. Janice Waller, the manager of the customer service department at First Bank of Jefferson County, can hire employees with a high school diploma for $20,000 annually and

employees with a bachelor's degree for \$30,000. She wants to maximize the number of customers served, given a fixed payroll. The following table shows how the total number of customers served varies with the number of employees:

Number of employees	Total number of customers served	
	High school diploma	Bachelor's degree
1	120	100
2	220	190
3	300	270
4	370	330
5	430	380
6	470	410

a. If Ms. Waller has a payroll of \$160,000, how should she allocate this budget in order to maximize the number of customers served?

b. If she has a budget of \$150,000 and currently hires three people with high school diplomas and three with bachelor's degrees, is she making the correct decision? Why or why not? If not, what should she do? (Assume she can hire part-time workers.)

c. If her budget is increased to \$240,000, how should she allocate this budget?

8. Bavarian Crystal Works designs and produces lead crystal wine decanters for export to international markets. The production manager of Bavarian Crystal Works estimates total and marginal production costs to be

$$TC = 10{,}000 + 40Q + 0.0025Q^2$$

and

$$MC = 40 + 0.005Q$$

where costs are measured in U.S. dollars and Q is the number of wine decanters produced annually. Because Bavarian Crystal Works is only one of many crystal producers in the world market, it can sell as many of the decanters as it wishes for \$70 apiece. Total and marginal revenue are

$$TR = 70Q \qquad \text{and} \qquad MR = 70$$

where revenues are measured in U.S. dollars and Q is annual decanter production.

a. What is the optimal level of production of wine decanters? What is the marginal revenue from the last wine decanter sold?

b. What are the total revenue, total cost, and net benefit (profit) from selling the optimal number of wine decanters?

c. At the optimal level of production of decanters, an extra decanter can be sold for \$70, thereby increasing total revenue by \$70. Why does the manager of this firm *not* produce and sell one more unit?

9. Joy Land Toys, a toy manufacturer, is experiencing quality problems on its assembly line. The marketing division estimates that each defective toy that leaves the plant costs the firm \$10, on average, for replacement or repair. The engineering department recommends hiring quality inspectors to sample for defective toys. In this way

many quality problems can be caught and prevented before shipping. After visiting other companies, a management team derives the following schedule showing the approximate number of defective toys that would be produced for several levels of inspection:

Number of inspectors	Average number of defective toys (per day)
0	92
1	62
2	42
3	27
4	17
5	10
6	5

The daily wage of inspectors is \$70.

a. How many inspectors should the firm hire?

b. What would your answer to *a* be if the wage rate is \$90?

c. What if the average cost of a defective toy is \$5 and the wage rate of inspectors is \$70?

MATHEMATICAL APPENDIX A Brief Presentation of Optimization Theory

Theory of Unconstrained Maximization

This section sets forth a mathematical analysis of unconstrained maximization. We begin with a single-variable problem in its most general form. An activity, the level of which is denoted as x, generates both benefits and costs. The total benefit function is $B(x)$ and the total cost function is $C(x)$. The objective is to maximize net benefit, NB, defined as the difference between total benefit and total cost. Net benefit is itself a function of the level of activity and can be expressed as

$$NB = NB(x) = B(x) - C(x) \tag{1}$$

The necessary condition for maximization of net benefit is that the derivative of NB with respect to x equal zero:

$$\frac{dNB(x)}{dx} = \frac{dB(x)}{dx} - \frac{dC(x)}{dx} = 0 \tag{2}$$

Equation (2) can then be solved for the optimal level of x, denoted x^*. Net benefit is maximized when

$$\frac{dB(x)}{dx} = \frac{dC(x)}{dx} \tag{3}$$

Since dB/dx is the change in total benefit with respect to the level of activity, this term is marginal benefit. Similarly for cost, dC/dx is the change in total cost with respect to the level of activity, and this term is marginal cost. Thus net benefit is maximized at the level of activity where marginal benefit equals marginal cost.

This unconstrained optimization problem can be easily expanded to more than one choice variable or kind of activity. To this end, let total benefit and total cost be functions of two different activities, denoted by x and y. The net benefit function with two activities is expressed as

$$NB = NB(x, y) = B(x, y) - C(x, y) \tag{4}$$

Maximization of net benefit when there are two activities affecting benefit and cost requires both of the partial derivatives of NB with respect to each of the activities to be equal to zero:

$$\frac{\partial NB(x, y)}{\partial x} = \frac{\partial B(x, y)}{\partial x} - \frac{\partial C(x, y)}{\partial x} = 0 \tag{5a}$$

$$\frac{\partial NB(x, y)}{\partial y} = \frac{\partial B(x, y)}{\partial y} - \frac{\partial C(x, y)}{\partial y} = 0 \tag{5b}$$

Equations (5a) and (5b) can be solved simultaneously for the optimal levels of the variables, x^* and y^*. Maximization of net benefit thus requires

(6a) $$\frac{\partial B}{\partial x} = \frac{\partial C}{\partial x}$$

and

(6b) $$\frac{\partial B}{\partial y} = \frac{\partial C}{\partial y}$$

For each activity, the marginal benefit of the activity equals the marginal cost of the activity. The problem can be expanded to any number of choice variables with the same results.

Turning now to a mathematical example, consider the following specific form of the total benefit and total cost functions:

(7) $$B(x) = ax - bx^2$$

and

(8) $$C(x) = cx - dx^2 + ex^3$$

where the parameters, a, b, c, d, and e are all positive.

Now the net benefit function can be expressed as

(9) $$NB = NB(x) = B(x) - C(x) = ax - bx^2 - cx + dx^2 - ex^3$$

To find the optimal value of x, take the derivative of the net benefit function with respect to x and set it equal to zero:

(10) $$\frac{dNB}{dx} = a - 2bx - c + 2dx - 3ex^2 = (a - c) - 2(b - d)x - 3ex^2 = 0$$

This quadratic equation can be solved using the quadratic formula or by factoring.[a]

Suppose the values of the parameters are $a = 60$, $b = 0.5$, $c = 24$, $d = 2$, and $e = 1$. The net benefit function is

(11) $$NB = NB(x) = 60x - 0.5x^2 - 24x + 2x^2 - x^3$$

Now take the derivative of NB [or substitute parameter values into equation (10)] to find the condition for optimization:

(12) $$(60 - 24) - 2(0.5 - 2)x - 3(1)x^2 = 36 + 3x - 3x^2 = 0$$

This equation can be factored: $(12 - 3x)(3 + x) = 0$. The solutions are $x = 4$, $x = -3$. (*Note:* The quadratic equation can also be used to find the solutions.) The value of x that maximizes net benefit is $x^* = 4$.[b] To find the optimal, or maximum, value of net benefit, substitute $x^* = 4$ into equation (11) to obtain

$$NB^* = 60(4) - 0.5(4)^2 - 24(4) + 2(4)^2 - (4)^3 = 104$$

Theory of Constrained Maximization

In a constrained maximization problem, a decision maker determines the level of the activities, or choice variables, in order to obtain the most benefit under a given cost constraint. In a constrained minimization problem, a decision maker determines the levels of the choice variables in order to obtain the lowest cost of achieving a given level of benefit. As we showed in the text, the solutions to the two types of problems are the same. We first consider constrained maximization.

Constrained maximization

We first assume a general total benefit function with two choice variables, the levels of which are denoted x and y: $B(x, y)$. The partial derivatives of this function represent the marginal benefit for each activity:

$$MB_x = \frac{\partial B(x, y)}{\partial x} \quad \text{and} \quad MB_y = \frac{\partial B(x, y)}{\partial y}$$

The constraint is that the total cost function must equal a specified level of cost, denoted as $\bar{C}$:

(13) $$C(x, y) = P_x x + P_y y = \bar{C}$$

where P_x and P_y are the prices of x and y. Now the Lagrangian function to be maximized can be written as

(14) $$\mathcal{L} = B(x, y) + \lambda(\bar{C} - P_x x - P_y y)$$

---

[a]The solution to a quadratic equation yields two values for x. The maximization, rather than minimization, solution is the value of x at which the second-order condition is met:

$$\frac{d^2NB}{dx^2} = -2(b - d) - 6ex < 0$$

[b]This value of x is the one that satisfies the second-order condition for a maximum in the preceding footnote:

$$\frac{d^2NB}{dx^2} = -2(-1.5) - 6(1)(4) = -21 < 0$$

where λ is the Lagrangian multiplier. The first-order condition for a maximum requires the partial derivatives of the Lagrangian with respect to the variables x, y, and λ to be zero:

$$\text{(14a)} \quad \frac{\partial \mathcal{L}}{\partial x} = \frac{\partial B}{\partial x} - \lambda P_x = 0$$

$$\text{(14b)} \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial B}{\partial y} - \lambda P_y = 0$$

$$\text{(14c)} \quad \frac{\partial \mathcal{L}}{\partial \lambda} = \overline{C} - P_x x - P_y y = 0$$

Notice that satisfaction of the first-order condition (14c) requires that the cost constraint be met.

Rearranging the first two equations (14a) and (14b):

$$\frac{\partial B}{\partial x} = \lambda P_x \quad \text{or} \quad \frac{MB_x}{P_x} = \lambda$$

$$\frac{\partial B}{\partial y} = \lambda P_y \quad \text{or} \quad \frac{MB_y}{P_y} = \lambda$$

It therefore follows that the levels of x and y must be chosen so that

$$\text{(15)} \quad \frac{MB_x}{P_x} = \frac{MB_y}{P_y}$$

The marginal benefits per dollar spent on the last units of x and y must be equal.

The three equations in (14) can be solved for the equilibrium values x^*, y^*, and λ^* by substitution or by Cramer's rule. Therefore, x^* and y^* give the values of the choice variables that yield the maximum benefit possible at the given level of cost.

Constrained minimization

For the constrained minimization problem, we want to choose the levels of two activities, x and y, to obtain a given level of benefit at the lowest possible cost. Therefore, the problem is to minimize $C = P_x x + P_y y$, subject to $\overline{B} = B(x, y)$, where $\overline{B}$ is the specified level of benefit. The Lagrangian function is

$$\text{(16)} \quad \mathcal{L} = P_x x + P_y y + \lambda[\overline{B} - B(x, y)]$$

The first-order conditions are

$$\text{(17)} \quad \begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= P_x - \lambda \frac{\partial B}{\partial x} = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= P_y - \lambda \frac{\partial B}{\partial y} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= [\overline{B} - B(x, y)] = 0 \end{aligned}$$

As in the constrained maximization problem, the first two equations can be rearranged to obtain

$$\text{(18)} \quad \begin{aligned} \frac{\partial B}{\partial x} &= \frac{1}{\lambda} P_x \quad \text{or} \quad \frac{MB_x}{P_x} = \frac{1}{\lambda} \\ \frac{\partial B}{\partial y} &= \frac{1}{\lambda} P_y \quad \text{or} \quad \frac{MB_y}{P_y} = \frac{1}{\lambda} \end{aligned}$$

Once again the marginal benefits per dollar spent on the last units of x and y must be the same, because, from (18),

$$\frac{MB_x}{P_x} = \frac{MB_y}{P_y}$$

The three equations in (17) can be solved for the equilibrium values x^*, y^*, and λ^* by substitution or by Cramer's rule. These are the values of the choice variables that attain the lowest cost of reaching the given level of benefit.

MATHEMATICAL EXERCISES

1. Assume the only choice variable is x. The total benefit function is $B(x) = 170x - x^2$, and the cost function is $C(x) = 100 - 10x + 2x^2$.
   a. What are the marginal benefit and marginal cost functions?
   b. Set up the net benefit function and then determine the level of x that maximizes net benefit.
   c. What is the maximum level of net benefit?

2. The only choice variable is x. The total benefit function is $B(x) = 100x - 2x^2$, and the total cost function is $C(x) = \frac{1}{3}x^3 - 6x^2 + 52x + 80$.
   a. What are the marginal benefit and marginal cost functions?
   b. Set up the net benefit function and then determine the level of x that maximizes net benefit. (Use the positive value of x.)
   c. What is the maximum level of net benefit?
3. A decision maker wishes to maximize total benefit, $B = 3x + xy + y$, subject to the cost constraint, $\overline{C} = 4x + 2y = 70$. Set up the Lagrangian and then determine the values of x and y at the maximum level of benefit, given the constraint. What are the maximum benefits?
4. A decision maker wishes to minimize the cost of producing a given level of total benefit, $\overline{B} = 288$. The cost function is $C = 6x + 3y$ and the total benefit function is $B = xy$. Set up the Lagrangian and then determine levels of x and y at the minimum level of cost. What is the minimum value of cost?
5. In Figure 3.1, the total benefit and total cost curves are represented by the following mathematical functions:

$$TB = TB(A) = 8A - 0.004A^2$$

and

$$TC = TC(A) = A + 0.006A^2$$

   a. Find the marginal benefit function. Verify that points c, b, and d in Figure 3.2 lie on the marginal benefit curve.
   b. Find the marginal cost function. Verify that points c', b, and d' in Figure 3.2 lie on the marginal cost curve.
   c. Derive the net benefit function. Verify the slopes of net benefit at points M, c'', and d'' in Figure 3.3.
   d. Find the optimal level of activity and the maximum value of net benefit. Does your answer match Figure 3.3?