2. Normal Random Variables

2.1 Continuous Random Variables

Whereas the possible values of the random variables considered in the previous chapter constituted sets of discrete values, there exist random variables whose set of possible values is instead a continuous region. These *continuous* random variables can take on any value within some interval. For example, such random variables as the time it takes to complete an assignment, or the weight of a randomly chosen individual, are usually considered to be continuous.

Every continuous random variable X has a function f associated with it. This function, called the *probability density function* of X, determines the probabilities associated with X in the following manner. For any numbers $a < b$, the area under f between a and b is equal to the probability that X assumes a value between a and b. That is,

$$P\{a \le X \le b\} = \text{area under } f \text{ between } a \text{ and } b.$$

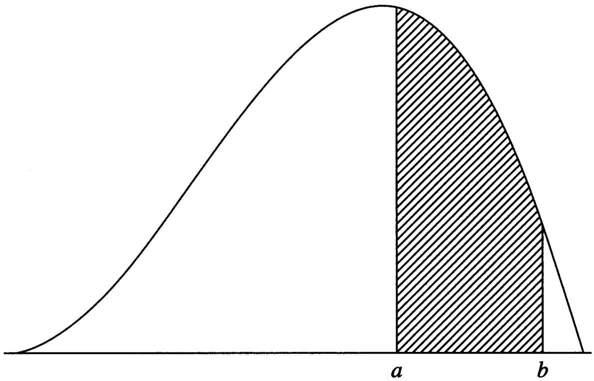
Figure 2.1 presents a probability density function.

2.2 Normal Random Variables

A very important type of continuous random variable is the normal random variable. The probability density function of a normal random variable X is determined by two parameters, denoted by μ and σ, and is given by the formula

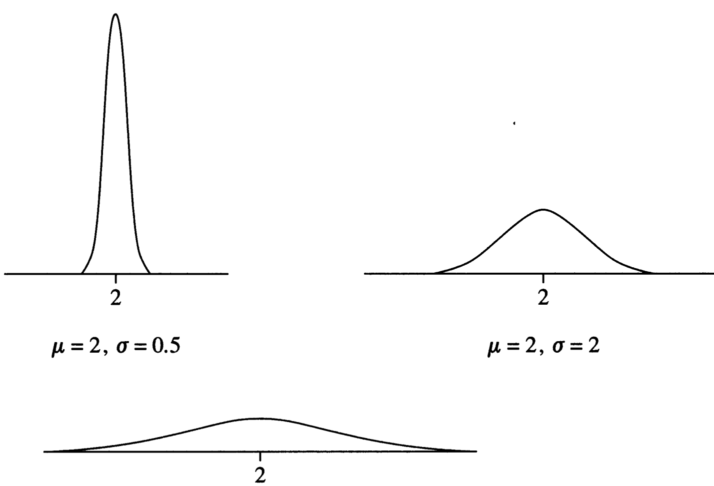
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$$

A plot of the normal probability density function gives a bell-shaped curve that is symmetric about the value μ, and with a variability that is measured by σ. The larger the value of σ, the more spread there is in f. Figure 2.2 presents three different normal probability density functions. Note how the curve flattens out as σ increases.



$P\{a \le X \le b\}$ = area of shaded region

Figure 2.1: Probability Density Function of X



$\mu = 2,\ \sigma = 0.5$

$\mu = 2,\ \sigma = 2$

$\mu = 2,\ \sigma = 4$

Figure 2.2: Three Normal Probability Density Functions

It can be shown that the parameters μ and σ^2 are equal to the expected value and to the variance of X, respectively. That is,

$$\mu = E[X], \qquad \sigma^2 = \mathrm{Var}(X).$$

A normal random variable having mean 0 and variance 1 is called a *standard normal* random variable. Let Z be a standard normal random variable. The function $\Phi(x)$, defined for all real numbers x by

$$\Phi(x) = P\{Z \le x\},$$

is called the *standard normal distribution function*. Thus $\Phi(x)$, the probability that a standard normal random variable is less than or equal to x, is equal to the area under the *standard normal density function*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty,$$

between $-\infty$ and x. Table 2.1 specifies values of $\Phi(x)$ when $x > 0$. Probabilities for negative x can be obtained by using the symmetry of the standard normal density about 0 to conclude (see Figure 2.3) that

$$P\{Z < -x\} = P\{Z > x\}$$

or, equivalently, that

$$\Phi(-x) = 1 - \Phi(x).$$

Example 2.2a Let Z be a standard normal random variable. For $a < b$, express $P\{a < Z \le b\}$ in terms of Φ.

Solution. Since

$$P\{Z \le b\} = P\{Z \le a\} + P\{a < Z \le b\},$$

we see that

$$P\{a < Z \le b\} = \Phi(b) - \Phi(a). \qquad \square$$

Example 2.2b Tabulated values of $\Phi(x)$ show that, to four decimal places,

$$P\{|Z| \le 1\} = P\{-1 \le Z \le 1\} = .6826,$$
$$P\{|Z| \le 2\} = P\{-2 \le Z \le 2\} = .9544,$$
$$P\{|Z| \le 3\} = P\{-3 \le Z \le 3\} = .9974. \qquad \square$$

Table 2.1: $\Phi(x) = P\{Z \leq x\}$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

When greater accuracy than that provided by Table 2.1 is needed, the following approximation to $\Phi(x)$, accurate to six decimal places, can be used: For $x > 0$,

$$\Phi(x) \approx 1 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2}(a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5),$$

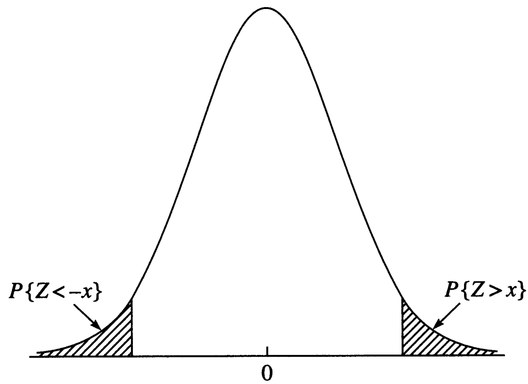


Figure 2.3: $P\{Z < -x\} = P\{Z > x\}$

where

$$y = \frac{1}{1 + .2316419x},$$
$$a_1 = .319381530,$$
$$a_2 = -.356563782,$$
$$a_3 = 1.781477937,$$
$$a_4 = -1.821255978,$$
$$a_5 = 1.330274429,$$

and

$$\Phi(-x) = 1 - \Phi(x).$$

2.3 Properties of Normal Random Variables

An important property of normal random variables is that if X is a normal random variable then so is $aX + b$, when a and b are constants. This property enables us to transform any normal random variable X into a standard normal random variable. For suppose X is normal with mean μ and variance σ^2. Then, since (from Equations (1.7) and (1.8))

$$Z = \frac{X - \mu}{\sigma}$$

has expected value 0 and variance 1, it follows that Z is a standard normal random variable. As a result, we can compute probabilities for any normal random variable in terms of the standard normal distribution function Φ.

Example 2.3a IQ examination scores for sixth-graders are normally distributed with mean value 100 and standard deviation 14.2. What is the probability that a randomly chosen sixth-grader has an IQ score greater than 130?

Solution. Let X be the score of a randomly chosen sixth-grader. Then,

$$\begin{aligned} P\{X > 130\} &= P\left\{\frac{X-100}{14.2} > \frac{130-100}{14.2}\right\} \\ &= P\left\{\frac{X-100}{14.2} > 2.113\right\} \\ &= 1 - \Phi(2.113) \\ &= .017. \end{aligned}$$ □

Example 2.3b Let X be a normal random variable with mean μ and standard deviation σ. Then, since

$$|X - \mu| \leq a\sigma$$

is equivalent to

$$\left|\frac{X-\mu}{\sigma}\right| \leq a,$$

it follows from Example 2.2b that 68.26% of the time a normal random variable will be within one standard deviation of its mean; 95.44% of the time it will be within two standard deviations of its mean; and 99.74% of the time it will be within three standard deviations of its mean. □

Another important property of normal random variables is that the sum of independent normal random variables is also a normal random variable. That is, if X_1 and X_2 are independent normal random variables with means μ_1 and μ_2 and with standard deviations σ_1 and σ_2, then $X_1 + X_2$ is normal with mean

$$E[X_1 + X_2] = E[X_1] + E[X_2] = \mu_1 + \mu_2$$

and variance

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = \sigma_1^2 + \sigma_2^2.$$

Example 2.3c The annual rainfall in Cleveland, Ohio, is normally distributed with mean 40.14 inches and standard deviation 8.7 inches. Find the probabiity that the sum of the next two years' rainfall exceeds 84 inches.

Solution. Let X_i denote the rainfall in year i $(i = 1, 2)$. Then, assuming that the rainfalls in successive years can be assumed to be independent, it follows that $X_1 + X_2$ is normal with mean 80.28 and variance $2(8.7)^2 = 151.38$. Therefore, with Z denoting a standard normal random variable,

$$\begin{aligned} P\{X_1 + X_2 > 84\} &= P\left\{Z > \frac{84 - 80.28}{\sqrt{151.38}}\right\} \\ &= P\{Z > .3023\} \\ &\approx .3812. \end{aligned}$$ □

The random variable Y is said to be a *lognormal* random variable with parameters μ and σ if $\log(Y)$ is a normal random variable with mean μ and variance σ^2. That is, Y is lognormal if it can be expressed as

$$Y = e^X,$$

where X is a normal random variable. The mean and variance of a lognormal random variable are as follows:

$$\begin{aligned} E[Y] &= e^{\mu + \sigma^2/2}, \\ \text{Var}(Y) &= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1). \end{aligned}$$

Example 2.3d Starting at some fixed time, let $S(n)$ denote the price of a certain security at the end of n additional weeks, $n \geq 1$. A popular model for the evolution of these prices assumes that the price ratios $S(n)/S(n-1)$ for $n \geq 1$ are independent and identically distributed (i.i.d.) lognormal random variables. Assuming this model, with lognormal parameters $\mu = .0165$ and $\sigma = .0730$, what is the probability that

(a) the price of the security increases over each of the next two weeks;
(b) the price at the end of two weeks is higher than it is today?

Solution. Let Z be a standard normal random variable. To solve part (a), we use that $\log(x)$ increases in x to conclude that $x > 1$ if and only if $\log(x) > \log(1) = 0$. As a result, we have

$$\begin{aligned} P\left\{\frac{S(1)}{S(0)} > 1\right\} &= P\left\{\log\left(\frac{S(1)}{S(0)}\right) > 0\right\} \\ &= P\left\{Z > \frac{-.0165}{.0730}\right\} \\ &= P\{Z > -.2260\} \\ &= P\{Z < .2260\} \\ &\approx .5894. \end{aligned}$$

Therefore, the probability that the price is up after one week is .5894. Since the successive price ratios are independent, the probability that the price increases over each of the next two weeks is $(.5894)^2 = .3474$.

To solve part (b), reason as follows:

$$\begin{aligned} P\left\{\frac{S(2)}{S(0)} > 1\right\} &= P\left\{\frac{S(2)}{S(1)}\frac{S(1)}{S(0)} > 1\right\} \\ &= P\left\{\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right) > 0\right\} \\ &= P\left\{Z > \frac{-.0330}{.0730\sqrt{2}}\right\} \\ &= P\{Z > -.31965\} \\ &= P\{Z < .31965\} \\ &\approx .6254, \end{aligned}$$

where we have used that $\log(\frac{S(2)}{S(1)}) + \log(\frac{S(1)}{S(0)})$, being the sum of independent normal random variables with a common mean .0165 and a common standard deviation .0730, is itself a normal random variable with mean .0330 and variance $2(.0730)^2$. □

2.4 The Central Limit Theorem

The ubiquity of normal random variables is explained by the central limit theorem, probably the most important theoretical result in probability.

This theorem states that the sum of a large number of independent random variables, all having the same probability distribution, will itself be approximately a normal random variable.

For a more precise statement of the central limit theorem, suppose that $X_1, X_2, \dots$ is a sequence of i.i.d. random variables, each with expected value μ and variance σ^2, and let

$$S_n = \sum_{i=1}^{n} X_i.$$

Central Limit Theorem *For large n, S_n will approximately be a normal random variable with expected value $n\mu$ and variance $n\sigma^2$. As a result, for any x we have*

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right\} \approx \Phi(x),$$

with the approximation becoming exact as n becomes larger and larger.

Suppose that X is a binomial random variable with parameters n and p. Since X represents the number of successes in n independent trials, each of which is a success with probability p, it can be expressed as

$$X = \sum_{i=1}^{n} X_i,$$

where X_i is 1 if trial i is a success and is 0 otherwise. Since (from Section 1.3)

$$E[X_i] = p \quad \text{and} \quad \mathrm{Var}(X_i) = p(1-p),$$

it follows from the central limit theorem that, when n is large, X will approximately have a normal distribution with mean np and variance $np(1-p)$.

Example 2.4a A fair coin is tossed 100 times. What is the probability that heads appears fewer than 40 times?

Solution. If X denotes the number of heads, then X is a binomial random variable with parameters $n = 100$ and $p = 1/2$. Since $np = 50$ we have $np(1-p) = 25$, and so

$$\begin{aligned}P\{X < 40\} &= P\left\{\frac{X-50}{\sqrt{25}} < \frac{40-50}{\sqrt{25}}\right\} \\ &= P\left\{\frac{X-50}{\sqrt{25}} < -2\right\} \\ &\approx \Phi(-2) \\ &= .0228.\end{aligned}$$

A computer program for computing binomial probabilities gives the exact solution .0176, and so the preceding is not quite as acccurate as we might like. However, we could improve the approximation by noting that, since X is an integral-valued random variable, the event that $X < 40$ is equivalent to the event that $X < 39 + c$ for any c, $0 < c \leq 1$. Consequently, a better approximation may be obtained by writing the desired probability as $P\{X < 39.5\}$. This gives

$$\begin{aligned}P\{X < 39.5\} &= P\left\{\frac{X-50}{\sqrt{25}} < \frac{39.5-50}{\sqrt{25}}\right\} \\ &= P\left\{\frac{X-50}{\sqrt{25}} < -2.1\right\} \\ &\approx \Phi(-2.1) \\ &= .0179,\end{aligned}$$

which is indeed a better approximation. □

2.5 Exercises

Exercise 2.1 For a standard normal random variable Z, find:

(a) $P\{Z < -.66\}$;
(b) $P\{|Z| < 1.64\}$;
(c) $P\{|Z| > 2.20\}$.

Exercise 2.2 Find the value of x when Z is a standard normal random variable and

$$P\{-2 < Z < -1\} = P\{1 < Z < x\}.$$

Exercise 2.3 Argue (a picture is acceptable) that

$$P\{|Z| > x\} = 2P\{Z > x\},$$

where $x > 0$ and Z is a standard normal random variable.

Exercise 2.4 Let X be a normal random variable having expected value μ and variance σ^2, and let $Y = a+bX$. Find values a, b ($a \neq 0$) that give Y the same distribution as X. Then, using these values, find $\text{Cov}(X, Y)$.

Exercise 2.5 The systolic blood pressure of male adults is normally distributed with a mean of 127.7 and a standard deviation of 19.2.

(a) Specify an interval in which the blood pressures of approximately 68% of the adult male population fall.
(b) Specify an interval in which the blood pressures of approximately 95% of the adult male population fall.
(c) Specify an interval in which the blood pressures of approximately 99.7% of the adult male population fall.

Exercise 2.6 Suppose that the amount of time that a certain battery functions is a normal random variable with mean 400 hours and standard deviation 50 hours. Suppose that an individual owns two such batteries, one of which is to be used as a spare to replace the other when it fails.

(a) What is the probability that the total life of the batteries will exceed 760 hours?
(b) What is the probability that the second battery will outlive the first by at least 25 hours?
(c) What is the probability that the longer-lasting battery will outlive the other by at least 25 hours?

Exercise 2.7 The time it takes to develop a photographic print is a random variable with mean 18 seconds and standard deviation 1 second. Approximate the probability that the total amount of time that it takes to process 100 prints is

(a) more than 1,710 seconds;
(b) between 1,690 and 1,710 seconds.

Exercise 2.8 Frequent fliers of a certain airline fly a random number of miles each year, having mean and standard deviation of 25,000 and 12,000 miles, respectively. If 30 such people are randomly chosen, approximate the probability that the average of their mileages for this year will

(a) exceed 25,000;
(b) be between 23,000 and 27,000.

Exercise 2.9 A model for the movement of a stock supposes that, if the present price of the stock is s, then – after one time period – it will either be us with probability p or ds with probability $1 - p$. Assuming that successive movements are independent, approximate the probability that the stock's price will be up at least 30% after the next 1,000 time periods if $u = 1.012$, $d = .990$, and $p = .52$.

Exercise 2.10 In each time period, a certain stock either goes down 1 with probability .39, remains the same with probability .20, or goes up 1 with probability .41. Asuming that the changes in successive time periods are independent, approximate the probability that, after 700 time periods, the stock will be up more than 10 from where it started.

REFERENCE

[1] Ross, S. M. (2010). *A First Course in Probability,* Prentice-Hall.